

1995

Beam Modes of Lasers with Misaligned Complex Optical Elements

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DISSERTATION APPROVAL

The abstract and dissertation of Anthony Alan Tovar for the Doctor of Philosophy in Electrical and Computer Engineering were presented April 21, 1995 and accepted by the dissertation committee and the doctoral program.

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ABSTRACT

An abstract of the dissertation of Anthony Alan Tovar for the Doctor of Philosophy in Electrical and Computer Engineering presented April 21, 1995.

Title: Beam Modes of Lasers with Misaligned Complex Optical Elements

A recurring theme in my research is that mathematical matrix methods may be used in a wide variety of physics and engineering applications. Transfer matrix techniques are conceptually and mathematically simple, and they encourage a systems approach. Once one is familiar with one transfer matrix method, it is straightforward to learn another, even if it is from a completely different branch of science. Thus it is useful to overview these methods, and this has been done here. Of special interest are the applications of these methods to laser optics, and matrix theorems concerning multipass optical systems and periodic optical systems have been generalized here to include, for example, the effect of misalignment on the performance of an optical system. In addition, a transfer matrix technique known as generalized beam method has been derived to treat misalignment effects in complex optical systems. Previous theories used numerical or ad hoc analytical solutions to a complicated diffraction integral. The generalized beam matrix formalism was also extended to higher-order beam modes of lasers and used to study mode discrimination in lasers with misaligned complex optical elements.

**BEAM MODES OF LASERS WITH
MISALIGNED COMPLEX OPTICAL ELEMENTS**

by

ANTHONY ALAN TOVAR

A dissertation in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY
in
ELECTRICAL AND COMPUTER ENGINEERING

Portland State University

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PREFACE

A man gives his friend, an apple farmer a mysterious new fruit he bought at the market. The man asks "What kind of apple is this?" The apple farmer briefly surveys the fruit and concludes "It's an orange-colored apple." Upon further inspection the apple farmer decides that it's a "coreless, orange-colored apple." Still not being completely satisfied the apple farmer cut open the fruit. After getting his hands sticky from the "apple" juice the farmer finally realized what he had - a "juicy, coreless, orange-colored apple."

The apple farmer's youngest son wanders by. Having never seen or heard of this bright-colored fruit he takes a bite, and declares "it looks like an orange."

Some percentage of science and engineering seems to be plagued with several maladies. As in the above paragraphs, outdated ideas, valid in their time, are often clung to too heavily. Interestingly, the originators of these ideas are often forgotten or mistaken. Another unfortunate occurrence is that people often apply complicated formalisms to problems which may be dealt with straightforwardly. One of the goals of this thesis is to, rectify some of these tendencies when they are applied to the fundamental problem of laser beam propagation and laser resonator mode design.

It is an impossible task to thank all of those whose efforts have made this work a reality. As a self-described natural philosopher, I am very cognizant of my utter dependence upon the contributions of my contemporaries and my predecessors. Thanks is due to, for example, Michael A. Faraday, James C. Maxwell, and Oliver Heaviside, for their role in the development of the electromagnetic theory used as a starting point here.

However, many other researchers are also responsible for developing the mathematics, physics, and inspiration required for this work. Certainly my parents have had an important role in the development of the this work. I would like to thank them not for "just having me," but for providing for my emotional and physical needs and creating an atmosphere that encouraged development. Without affirmation, friendship, and love, it is impossible for one to develop and mature, and without this process, this thesis would never have been written. The world is a place of interdependent relationships, and this synergy has played an important role in the development of this work.

Though a great many people deserve some measure of thanks for their contribution to this dissertation, there are several who should be mentioned by name. First and foremost, loving thanks to my wife, Lyn, for all the support and hard work, often performing the duties of two parents while I was laboring over the project. Thanks too, to my sons Forrest and Sylvan who help me to see what life is really all about. From the electrical engineering department, I would like to thank the office coordinator Shirley C. Clark, and Laura B. Riddell and Ellen E. Wack of the secretarial staff. They were excellent guides in the university's bureaucratic jungle. I have benefitted from my interactions with the laser group, who I would also like to thank.

Finally, I would like to thank my collaborator/mentor/role-model/hero Dr. Lee W. Casperson. His patience and support have helped me in all aspects of "being and becoming" a scientist. Even in the early days of my doctoral research, he seemed satisfied with my ability to do work. However, many projects were left uncompleted, and he would often say "work, finish, publish" quoting Michael Faraday who thought it so important that he had this put on his tombstone (the quote originated from Benjamin Franklin). Since then Dr. Casperson and I have collaborated on several publications, and it is hoped that my modest improvement has given him some small measure of

satisfaction. One of my life's goals is to some day inspire others as Dr. Casperson has done me. This would be the ultimate tribute to a great man.

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CHAPTER I

INTRODUCTION

A laser is essentially an optical gain medium in a resonator structure which provides positive feedback. A basic requirement for laser operation is that the amplifier gain be greater than the loss from the resonator. Small misalignments often cause a dramatic decrease of the feedback in the resonator. Thus, laser alignment is an important practical problem for laser designers and users.

In the simplest confined-mode laser designs, slight misalignments only redefine the optic axis. Because of this and since the assumption of perfect alignment leads to a dramatic simplification of the equations governing laser beam propagation, the effects of misalignment are usually ignored. In the laboratory and on the factory floor, lasers are aligned to achieve maximum power. However, in modern laser design, maximum power may occur at a different alignment from that which produces the optimum spectrum, beam quality, or pulsewidth. Thus, there may not always be a unique definition of "aligned."

A novel transfer matrix method is developed here to determine quantitatively the effect of misalignment on laser performance. Transfer matrix techniques are conceptually and mathematically simple, and they encourage a systems approach. Once one is familiar with one transfer matrix method, it is straightforward to learn another, even if it is from a completely different branch of science. The merits of our emphasis on simplicity has been echoed by scientists throughout the ages. Andre Weil put it bluntly "In the future, as in the past, the great ideas must be simplifying ideas." Josiah W. Gibbs (1839-1903) said "One of the principal objects of theoretical research in my department of

knowledge is to find the point of view from which the subject appears in its greatest simplicity." This work is different from much of the literature on the subject which uses unnecessarily complicated mathematical formalisms. Another difficulty inherent to the modern publication process also exists in the literature. As may be seen from, for example, the work of Lee W. Casperson, even if articles are well written, they are scattered throughout a variety of journals [1]-[139]. In addition to generalizing Gaussian laser beam theory to include misaligned complex optical systems and laser resonators, this thesis provides a much needed summary of Gaussian laser beam theory for isotropic linear media.

The branch of laser beam theory developed in this thesis is based on what we call "Generalized beam matrices." This theory is one of a class of transfer matrix methods. Chapter II contains a novel summary of the properties and theorems common to 2×2 transfer matrices. These properties have important ramifications in laser beam theory.

In Chapter III, two important transfer matrix theorems are discussed in detail. The reverse theorem governs light propagation through an optical system in the backward direction, and Sylvester's theorem governs light propagation through periodic optical systems. For every 2×2 transfer matrix method, we show that there exists a new augmented 3×3 transfer matrix method. The use of the 3×3 augmented matrices often generalizes 2×2 matrix methods in an intuitive manner. For example, transfer matrix analysis of circuits with intranetwork independent voltage and current sources may be performed using augmented matrices. The reverse theorem and Sylvester's theorem are also generalized in Chapter III for optical systems represented by these augmented matrix methods.

Laser electromagnetics in the subject of Chapter IV. After a novel summary of Maxwell's equations, an important wave equation is derived. To apply the wave equation to a laser medium, knowledge of the laser's gain and dispersion properties must be

known. These gain and refractive index formulas are summarized for different types of laser media. This Chapter starts from first principles and lays the ground work for the succeeding chapters.

In Chapter V, the generalized beam matrix method is developed to propagate off-axis Gaussian beams in astigmatic optical systems that may include tilted, displaced, or curved optical elements. Unlike a previous generalized ray matrix formalism, optical elements that possess gain or loss such as Gaussian apertures, complex lenslike media, and amplifiers are included; and new beam transformations are obtained. In addition, more new beam transformations are obtained that extend the generalized beam matrix methods to polynomial-Gaussian beams.

The generalized beam matrix method is used in Chapter VI to investigate the mode selection of astigmatic misaligned optical systems with loss or gain. In these optical systems, the usual real argument polynomial-Gaussian beams are not eigenfunctions, and off-axis complex argument polynomial beams must be used. New beam transformations for these complex-argument modes are found. Stability criteria are developed and mode selection in lasers and periodic misaligned complex optical systems is discussed.

CHAPTER II

2 x 2 TRANSFER MATRIX THEORY

INTRODUCTION

An increasing portion of optics and physics in general is easily dealt with using simple transfer matrix methods. These matrix methods may be used to, for example, trace laser beams through optical systems and resonators.

Some of the advantages of the transfer matrix approach are discussed by A. Gerrard & J. M. Burch in the preface to their book Introduction To Matrix Methods In Optics [140]

Our purpose in writing this book has been ... to encourage the adoption of simple matrix methods in the teaching of optics at the undergraduate and technical college level. Many of these methods have been known for some time but have not found general acceptance; we believe that the time has now come for lecturers to reconsider their value. We believe this partly because the use of matrices is now being taught quite widely in schools, and many students will already have glimpsed something of the economy and elegance with which, for a linear system, a whole wealth of input-output relations can be expressed by a single matrix.

"A second reason is that, for more than a decade, the field of optics has been enriched enormously by contributions from other disciplines such as microwave physics and electrical engineering. Although an engineering student may be a newcomer to optics, he may well have encountered matrix methods during his lectures on electrical filters or transmission lines; we think he will welcome an optics course which, instead of barricading itself behind its own time-honored concepts, links itself recognizably to other disciplines.

Because of these unique advantages, it is useful to examine transfer matrix optical methods as a group.

In this chapter, a unified overview of the general properties of these transfer matrix methods is given. After a brief summary of the essentials of transfer matrix mathematics,

several different transfer matrix methods are discussed. A variety of matrix theorems and invariance transformations are examined, and associated matrix methods are discussed.

TRANSFER MATRIX MATHEMATICS

The propagation formulas for a given 2x2 transfer matrix theory can be written in the form

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \quad (2.1)$$

where X and Y are the two dependent parameters that change due to the existence of the system. In our notation, whether the signal is injected in the forward or reverse direction, the "2" subscripts represent the "output" and the "1" subscripts represent the "input" parameters. The A and D matrix elements are dimensionless. The units of B are the units of X divided by the units of Y , and the units of C are the multiplicative inverse of the units of B . The determinant of the matrix is $AD - BC$. As an alternate notation, Eq. (2.1) is sometimes written

$$\begin{bmatrix} X \\ Y \end{bmatrix}_2 = \begin{bmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}_1. \quad (2.2)$$

The ABCD matrix in Eq. (2.1) may represent forward propagation through a single system element, or it may refer to an overall system matrix. To obtain a system matrix, one need only multiply the matrix representations for each of the individual system elements in reverse order. As an example, the special case of a system consisting of two cascaded elements is considered. If the first element's transfer characteristic is given in Eq. (2.1), and the second is characterized by the matrix equation

$$\begin{bmatrix} X_3 \\ Y_3 \end{bmatrix} = \begin{bmatrix} A_{23} & B_{23} \\ C_{23} & D_{23} \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}, \quad (2.3)$$

then it may be seen by substituting Eq. (2.1) into Eq. (2.3) that the total system matrix is

$$\begin{bmatrix} X_3 \\ Y_3 \end{bmatrix} = \begin{bmatrix} A_{23} & B_{23} \\ C_{23} & D_{23} \end{bmatrix} \begin{bmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} . \quad (2.4)$$

Thus, for a two-element system, the system matrix is the product of the individual element matrices in reverse order. It follows by induction that if the system consists of n optical elements, then the total system matrix is

$$T_{system} = T_n T_{n-1} \cdots T_3 T_2 T_1 . \quad (2.5)$$

In many matrix theories, the determinant of each of the system element matrices is unity. Since the determinant of a product is the product of the determinants, it follows that, for such a matrix theory, any system matrix will be unimodular (i.e. unit determinant). For nonunimodular matrix theories, the determinant usually carries important information.

TRANSFER MATRIX PHYSICS

Transfer matrix methods may be applied to a wide variety of disciplines in physics. In this section, a few of the matrix methods which are of particular interest to laser physicists and engineers are briefly discussed.

Paraxial Ray Matrices

One of the most fundamental problems in optics involves tracing light rays through optical systems. In the context of laser optics, these light rays propagate at a small angle from the optic axis. In this case, one may define a matrix method [141]. This ray matrix method has a number of advantages over the conventional geometric optics approach to light ray propagation [142].

For paraxial ray matrices, the signal vector components represent the position and slope of a light ray, and the matrix method may be written

$$\begin{bmatrix} r_x \\ r_x' \end{bmatrix}_2 = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} r_x \\ r_x' \end{bmatrix}_1 , \quad (2.6)$$

where the matrix elements are purely real. For astigmatic optical systems, there is a

similar equation with a y -subscript. If the input ray and the output ray are in media with the same index of refraction, then the determinant of the matrix is unity. A lens of focal length f_x and a length, d , of freespace are represented by the matrices

$$T_{lens} = \begin{bmatrix} 1 & 0 \\ -f_x^{-1} & 1 \end{bmatrix}, \quad (2.7)$$

$$T_{freespace} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}, \quad (2.8)$$

respectively. Mode confinement in laser resonators, light propagation in lens-waveguides, and chromatic aberration in refracting telescopes are a few of the many problems that may be approached systematically with the aid of Eqs. (2.7) and (2.8). There are also paraxial ray matrix representations for many other optical elements. This flexibility provides access to many other design possibilities. In addition, there exists a transfer matrix method to trace aberrated light rays in nonparaxial optical systems [143]. This ability makes it possible to design, for example, camera lenses using transfer matrix methods.

Gaussian Beam Matrices

Laser beams always have a finite width which is associated with the transverse field distribution. For most conventional lasers this transverse distribution is Gaussian. Gaussian beam propagation may be treated with transfer matrix methods, and because of the simplicity of these methods, this is almost always done. The 2×2 Gaussian beam matrix method was created by H. Kogelnik [144]-[146], and other important early work was performed by J. Arnaud [147]-[154].

The Gaussian beam matrix formalism may be written

$$\begin{bmatrix} u_x \\ (1/q_x)u_x \end{bmatrix}_2 = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} u_x \\ (1/q_x)u_x \end{bmatrix}_1, \quad (2.9)$$

where u_x does not have any important meaning. The beam parameter q_x is related to the

1/e beam amplitude width, w_x , and radius of curvature of the phase fronts, R_x , of the Gaussian beam by the relation

$$\frac{1}{q_x} \equiv \frac{1}{R_x} - i \frac{\lambda}{\pi n w_x^2} . \quad (2.10)$$

The vacuum wavelength of the beam is λ and the index of refraction of the medium at which q_x is being evaluated is n . The x -subscripts are a reminder that similar results apply to the y -direction, and therefore the Gaussian beam may be elliptical.

Similar to ray matrices, Eqs. (2.7) and (2.8) are used to represent lenses and freespace, respectively. Indeed, most optical elements representable by a paraxial ray matrix have the same Gaussian beam matrix representation. However, while paraxial ray matrices are strictly real, Gaussian beam matrices may be complex. Thus, there are additional optical elements representable by Gaussian beam matrices. An example of a "complex" optical element is the Gaussian aperture [22] [153], [155]-[159] which is represented by the following matrix:

$$T_{\text{Gaussian aperture}} = \begin{bmatrix} 1 & 0 \\ -i\lambda/(\pi w_x^2) & 1 \end{bmatrix} . \quad (2.11)$$

Gaussian apertures are sometimes used in high-diffraction-loss resonators which are used for high power lasers.

Electric Circuit Matrices

The most fundamental and frequently encountered discipline in electrical engineering is circuit analysis. Linear circuit analysis may be performed using transfer matrix methods. The current and voltage at an output port (V_2, I_2) are related to the voltage and current at the input port (V_1, I_1) by

$$\begin{bmatrix} V \\ I \end{bmatrix}_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix}_1 . \quad (2.12)$$

Resistors ($Z = R$), capacitors ($Z = -i/(\omega C)$), and inductors ($Z = i\omega L$), are represented

as impedances. These impedances may be in series or in parallel with the network:

$$T_{series \text{ impedance}} = \begin{bmatrix} 1 & -Z_{series} \\ 0 & 1 \end{bmatrix}, \quad (2.13)$$

$$T_{parallel \text{ impedance}} = \begin{bmatrix} 1 & 0 \\ -Z_{parallel}^{-1} & 1 \end{bmatrix}. \quad (2.14)$$

Operational amplifiers, transmission lines, and transistors (operating in the linear region) may also be analyzed with this matrix method. Microwave circuit engineers often use this, or similar matrix methods, in the design of microwave transistor amplifiers [160].

Jones Polarization Matrices

Transfer matrix methods are the dominant theoretical tools used in the field of polarization optics. If the two orthogonal components of a light beam's polarization are E_x and E_y , then the Jones calculus matrix method may be written [161]-[172]

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix}_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}_1. \quad (2.15)$$

An important optical element is the waveplate, whose matrix is

$$T_{waveplate} = e^{-i\phi} \begin{bmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{bmatrix}, \quad (2.16)$$

where Γ is the phase retardation, and ϕ is the absolute phase change. These parameters are related to the length of the medium and the index of refraction for polarization parallel to the eigenaxes by $\Gamma = (n_s - n_f)\omega l/c$, and $\phi = 1/2(n_s + n_f)\omega l/c$. Often the eigenaxes of the waveplate are rotated by an angle θ with respect to the global axis, and the rotation matrix

$$T_{rotation} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad (2.17)$$

is of interest. In this case, the matrix for a rotated waveplate becomes

$$T_{rotated \text{ waveplate}} = e^{-i\phi} \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \quad (2.18)$$

Narrow band optical filters are designed using rotated waveplates [173]. In addition to waveplates, polarizers, media with optical activity, Faraday rotators, and linear and quadratic electro-optic media also have matrix representations.

Feedback Matrices

Fresnel reflection occurs when a light beam encounters a dielectric boundary at normal incidence. In addition to being reflected, part of the beam is transmitted through the boundary. This is problematic since dielectric media have at least two sides, and hence two or more boundaries. Part of the incident beam bounces back and forth between the two boundaries several times before exiting the medium. In this way, the total reflected light signal is an infinite sum of smaller components, and the corresponding calculations become tedious, if not intractable.

An elementary transfer matrix method exists which simplifies the mathematics considerably. If E^+ and E^- represent the complex amplitude of the time independent electric field propagating in the $+z$ and $-z$ directions, respectively, then

$$\begin{bmatrix} E^+ \\ E^- \end{bmatrix}_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E^+ \\ E^- \end{bmatrix}_1. \quad (2.19)$$

The matrix elements are related to conventional scattering matrix amplitude reflection and transmission coefficients via

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -(r_{11}r_{22} - t_{12}t_{21})/t_{12} & r_{22}/t_{12} \\ -r_{11}/t_{12} & 1/t_{12} \end{bmatrix}. \quad (2.20)$$

The matrix representation for a dielectric boundary is

$$T_{\text{dielectric boundary}} = \begin{bmatrix} 1/t & -r/t \\ -r/t & 1/t \end{bmatrix}, \quad (2.21)$$

where $r \equiv (n_1 - n_2)/(n_1 + n_2)$, $t \equiv 2(n_1 n_2)^{1/2}/(n_1 + n_2)$, are the reflection and transmission coefficients, respectively. The matrix representation for a dielectric medium is

$$T_{\text{dielectric medium}} = \begin{bmatrix} \exp(-i\beta_o l) & 0 \\ 0 & \exp(i\beta_o l) \end{bmatrix} \quad (2.22)$$

where $\beta_o \equiv 2\pi v n(v)/c$. As an example, the system matrix for a dielectric medium with two boundaries is

$$T_{etalon} = \begin{bmatrix} 1/t & r/t \\ r/t & 1/t \end{bmatrix} \begin{bmatrix} \exp(-i\beta_o l) & 0 \\ 0 & \exp(i\beta_o l) \end{bmatrix} \begin{bmatrix} 1/t & -r/t \\ -r/t & 1/t \end{bmatrix} \quad (2.23)$$

where $\beta_o = 2\pi v n_2/c$, and n_2 is the refractive index of the dielectric medium. From Eq. (2.19) it may be shown that the intensity transmission of an optical system represented by feedback matrices is

$$Intensity\ Transmission \equiv \frac{I_2^+}{I_1^+} \bigg|_{I_2^- = 0} = \frac{(AD - BC)^*(AD - BC)}{D^*D} \quad (2.24)$$

where the intensity is proportional to E^*E . Thin film reflection and antireflection coatings as well as optical filters may be designed using these dielectric boundary and dielectric medium matrices.

TRANSFER MATRIX ANALOGIES

In the study of matrix optics, one finds that there are individual matrices which commonly arise in several matrix theories. To reinforce quantitative analogies between systems, it is useful to emphasize these matrices.

The first matrix to be considered is the identity matrix,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (2.25)$$

which changes neither the X component nor the Y component of the signal. Thus, the identity matrix may be used as a continuity condition. However, it is also sometimes desirable to design an overall system which does not change the input signal. In this case the system matrix has this identity matrix form. This design methodology is of interest in a Gaussian beam optical system when, for example, a flat untilted mirror is optimum but practical problems force one to position the mirror away from the end of the laser [59]. In the Jones Calculus matrix method for optical polarization calculations, one can

imagine an optical system which due to its nature distorts the input polarization. In this case, it may be desired to synthesize a system so that the overall system matrix is the identity matrix.

The unimodular matrix

$$\begin{bmatrix} 1 & \chi \\ 0 & 1 \end{bmatrix} \quad (2.26)$$

changes the X signal component without changing the Y signal component. If χ is real and positive here, then expression (2.26) is the matrix representation of a uniform medium in both the paraxial ray and Gaussian beam matrix theory [Eq. (2.8)]. If χ is complex, then expression (2.26) represents an amplifier or absorber in the Gaussian beam theory only [134]. In electrical circuit matrix theory, χ represents a series impedance [Eq. (2.13)].

A dual of expression (2.26) is the unimodular matrix

$$\begin{bmatrix} 1 & 0 \\ \chi & 1 \end{bmatrix} \quad (2.27)$$

which changes the Y signal component without changing the X signal component. It is used to represent a thin lens [Eq. (2.7)] and/or a Gaussian aperture [Eq. (2.11)] in the Gaussian beam theory, and a shunt impedance in the electrical theory [Eq. (2.14)].

Due to the pervasiveness and importance of the matrices in expressions (2.25)-(2.27), they are viable candidates as matrix primitives from which an arbitrary system may be synthesized [59]. In the paraxial ray and Gaussian beam theories, many simple systems are made up of flat mirrors [expression (2.25)], uniform media [expression (2.26)], and lenses [expression (2.27)]. Similarly, in the electric circuit theory, many two-port systems are composed only of series [expression (2.26)] and shunt [expression (2.27)] impedances.

Scaling can be obtained by the not-necessarily-unimodular matrix

$$\begin{bmatrix} \chi_X & 0 \\ 0 & \chi_Y \end{bmatrix} \quad (2.28)$$

where χ_X and χ_Y are the scale factors for the X and Y signal components respectively. Expression (2.28) represents an anisotropic medium in Jones' calculus [Eq. (2.16)] and a dielectric medium in the feedback matrix method [Eq. (2.22)]. In the Gaussian beam theory, expression (2.28) with $\chi_X = 1$ represents a dielectric or gain boundary [134]. In the unimodular limit, $\chi_X = \chi_Y^{-1}$, and a matrix of this form is used to represent an ideal transformer in the electrical theory.

Symmetry is often considered a desirable property in a system. For our purposes, a symmetric system is one which causes a signal injected backward to undergo the same transformation as one injected in the forward direction. As will be seen, for several types of unimodular systems, the requirement of symmetry implies that $A = D$. A unimodular matrix in which the diagonal elements are equal ($A = D$) can be put in the form

$$\begin{bmatrix} \cos\theta & \chi\sin\theta \\ -\chi^{-1}\sin\theta & \cos\theta \end{bmatrix} \quad (2.29)$$

where the potentially complex χ and θ are defined by the relationships: $A = D \equiv \cos\theta$, $B \equiv -C\chi^2$. In many 2x2 transfer matrix theories, there are also individual elements that are represented by a matrix of this form. In particular, in the Gaussian beam theory, expression (2.29) is used to represent a complex lenslike medium [1] [14] [145]. Similarly, the matrix expression (2.29) is also used to represent a transmission line in the electrical circuit theory. Thus it follows that a symmetric Gaussian beam optical system can be synthesized with a single complex lenslike medium and a symmetric electrical system can be synthesized with a single transmission line. This matrix expression (2.29) is common in matrix theories derived from a second order differential equation with constant coefficients. If the coefficients are nonconstant, then alternate solutions to the differential equations are of interest [79] [81].

An important special case of expression (2.29) occurs when $\chi = 1$ and θ is real:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \quad (2.30)$$

This matrix represents a pure rotation about the origin in the XY plane. Such rotation matrices are prevalent in Jones calculus calculations [see, for example, Eq. (2.18)].

Other operations in the XY plane include the nonunimodular matrices for mirror reflection across the X axis,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (2.31)$$

and the Y axis,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (2.32)$$

The matrix for mirror reflection across both X and Y signal component axes is also a special case of expression (2.28). It is written as

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (2.33)$$

and it occurs in several unimodular matrix theories. In both the paraxial ray and Gaussian beam theories, expression (2.33) represents a retroreflecting mirror. If the analogy with paraxial ray theory is exploited, then it is suggested that this matrix can be used to represent cross-wiring in the electrical circuit theory. In the paraxial ray theory, the matrix (2.31) is used to represent a phase conjugate mirror.

TRANSFER MATRIX THEOREMS

One of the advantages of studying transfer matrix methods in parallel is that matrix theorems learned in one area may be readily applied to another. It is therefore useful to examine these theorems systematically.

Sylvester's Theorem

In addition to matrix multiplication of individual matrices, there are several other

meaningful operations that may be performed on individual and system matrices. A first step in the synthesis process may include the interpretation of these operations. The first operation considered is represented by the s th power of a 2x2 matrix [174], which is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \frac{\tau^{(s-1)/2}}{\sin\theta'} \begin{bmatrix} A \sin(s\theta') - \tau^{1/2} \sin[(s-1)\theta'] & B \sin(s\theta') \\ C \sin(s\theta') & D \sin(s\theta') - \tau^{1/2} \sin[(s-1)\theta'] \end{bmatrix}, \quad (2.34)$$

where

$$\cos\theta' \equiv (A + D)\tau^{-1/2}/2. \quad (2.35)$$

This is the 2x2 special case of Sylvester's matrix polynomial theorem [175]. In the unimodular limit $AD - BC = 1$, and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \frac{1}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \sin[(s-1)\theta] & B \sin(s\theta) \\ C \sin(s\theta) & D \sin(s\theta) - \sin[(s-1)\theta] \end{bmatrix}, \quad (2.36)$$

where

$$\cos\theta \equiv \frac{A + D}{2}. \quad (2.37)$$

Not only is Sylvester's theorem valid for positive integer powers of matrices, but for negative integer powers and roots as well [133] [174]. However, this corresponds to a potential design criterion. Suppose that it is desired to synthesize a known system matrix as the cascade of s identical subsystems. The design procedure amounts to taking the s th root ($1/s$ power) of the system matrix, and writing it in terms of some set of defined matrix primitives. Sylvester's theorem is also used directly in the analysis of specific periodic systems.

Reverse Theorems

Just as the matrix operation that corresponds to Sylvester's theorem has the physical interpretation of the cascade of s identical optical systems, other matrix operations can also be interpreted. It can be seen from Eq. (2.1) that a given system matrix yields the output signal given an input signal. Multiplying both sides of Eq. (2.1) by the inverse of

the system matrix, it follows that the matrix inverse can be interpreted as the matrix that yields the input given the output. A reverse matrix may be defined as a matrix that yields the "input signal propagating in the reverse direction" given the "output signal propagating in the reverse direction." Similarly, the inverse of a reverse matrix yields the output going in the reverse direction given the input going in the reverse direction.

In this way, the reverse matrix is the appropriate matrix for propagating through a system backward. When a system matrix is equal to its own reverse, the system is (by our definition) symmetric. For our purposes, this symmetry may be part of the given design criteria. However, different matrix theories may possess different reverse matrix forms. For paraxial light rays and Gaussian beams in first-order optical systems, and electrical signals in two-port networks, the reverse matrix is [131]

$$T_R = \begin{bmatrix} D & B \\ C & A \end{bmatrix} . \quad (2.38)$$

As will be shown, this is also the reverse matrix for other unimodular matrix theories in which the Y parameter is the derivative of the X parameter. If a given matrix is equal to its reverse matrix, then the system is symmetric, and from Eq. (2.38) it follows that the condition for symmetry is $A = D$.

As opposed to these matrix theories, the reverse matrix for the unitary form of Jones calculus is [176]

$$T_R = \begin{bmatrix} A & C \\ B & D \end{bmatrix} . \quad (2.39)$$

Thus a birefringent optical system is symmetric if its Jones matrix elements have the property that $B = C$.

Matrix Factorization

Design criteria for a given system can be realized as constraints on the system matrix. These constraints may be used to obtain the system matrix. Once the system

matrix is known, then it is of interest to consider procedures to determine the optical components needed to fulfill these criteria. This is accomplished by factoring the system matrix into matrix primitives. Each of these primitives represents an optical component available to the optical designer.

As part of the design criteria for a given periodic system, the input signal may be required to repeat after propagating through s identical subsystems. Indeed, the sinusoidal nature of Sylvester's Theorem Eq. (2.36) suggests such a repetitive signal condition. In particular, if

$$T^s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.40)$$

then the input signal is reproduced after propagating through s systems, or through a single system s times. It may be seen from Eq. (2.36) that this occurs when $s\theta = 2k\pi$ where k is an integer. In terms of matrix elements, the repetitive signal condition from Eq. (2.37) is

$$\frac{A+D}{2} = \cos(2k\pi/s) \quad (2.41)$$

where the restriction

$$0 \leq k \leq s/2 \quad (2.42)$$

is made to avoid duplicating solutions. A graphical interpretation of the result has been used in the design of annular gain lasers [58].

When the system matrix, based on design criteria is known, it is required to factor the system matrix in terms of matrix primitives. Each of these matrix primitives must represent a manufacturable optical component. If Eqs. (2.26) and (2.27) along with Eq. (2.28) with $\chi_x = 1$ are used as the matrix primitives, then there are eight possible factorizations [59]:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \begin{bmatrix} 1 & (A-1)(AD-BC)/C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C/(AD-BC) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & B+(1-A)D/C \\ 0 & 1 \end{bmatrix}$$

$$(2.43)$$

$$= \begin{bmatrix} 1 & (A-1)/C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \begin{bmatrix} C/(AD-BC) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & B+(1-A)D/C \\ 0 & 1 \end{bmatrix} \quad (2.44)$$

$$= \begin{bmatrix} 1 & (A-1)/C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \begin{bmatrix} 1 & B+(1-A)D/C \\ 0 & 1 \end{bmatrix} \quad (2.45)$$

$$= \begin{bmatrix} 1 & (A-1)/C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & [B+(1-A)D/C]/(AD-BC) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \quad (2.46)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \begin{bmatrix} [C+(1-A)D/B]/(AD-BC) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A-1)/B & 1 \end{bmatrix} \quad (2.47)$$

$$= \begin{bmatrix} C+(1-A)D/B & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A-1)/B & 1 \end{bmatrix} \quad (2.48)$$

$$= \begin{bmatrix} C+(1-A)D/B & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & B/(AD-BC) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A-1)/B & 1 \end{bmatrix} \quad (2.49)$$

$$= \begin{bmatrix} C+(1-A)D/B & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & B/(AD-BC) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A-1)(AD-BC)/B & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & AD-BC \end{bmatrix} \quad (2.50)$$

If the system matrix is constrained to be unimodular, then these eight factorizations reduce to two:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & (A-1)/C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & (D-1)/C \\ 0 & 1 \end{bmatrix}, \quad (2.51)$$

$$= \begin{bmatrix} 1 & 0 \\ (D-1)/B & 1 \end{bmatrix} \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A-1)/B & 1 \end{bmatrix}. \quad (2.52)$$

Of course, the factorization in Eq. (2.51) is valid only when C is nonzero, and the factorization in Eq. (2.52) is invalid unless B is nonzero. If the system matrix elements B and C are both zero, then the additional factorizations [59]

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & -\alpha D \\ 0 & D \end{bmatrix} \quad (2.53)$$

$$= \begin{bmatrix} A & -\alpha A \\ 0 & D \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \quad (2.54)$$

$$= \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ -\beta A & D \end{bmatrix} \quad (2.55)$$

$$= \begin{bmatrix} A & 0 \\ -\beta D & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} \quad (2.56)$$

are necessary.

If in addition to the matrix primitive expressions (2.26) and (2.27), optical components represented by expression (2.29) are available, the the following two-matrix factorizations may be of interest:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & (A-D)/C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D & (D^2-1)/C \\ C & D \end{bmatrix} \quad (2.57)$$

$$= \begin{bmatrix} A & (A^2-1)/C \\ C & A \end{bmatrix} \begin{bmatrix} 1 & (D-A)/C \\ 0 & 1 \end{bmatrix} \quad (2.58)$$

$$= \begin{bmatrix} 1 & 0 \\ (D-A)/B & 1 \end{bmatrix} \begin{bmatrix} A & B \\ (A^2-1)/B & A \end{bmatrix} \quad (2.59)$$

$$= \begin{bmatrix} D & B \\ (D^2-1)/B & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A-D)/B & 1 \end{bmatrix} . \quad (2.60)$$

In the Jones calculus method, other matrix primitives are of interest [162].

As previously suggested, there exist design criteria that demand the output signal be identical to the input signal when the system is periodic. Thus it is useful to consider other factorizations of the identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^2 \quad (2.61)$$

$$= \begin{bmatrix} \cos\phi & \gamma\sin\phi \\ \gamma^{-1}\sin\phi & -\cos\phi \end{bmatrix}^2 \quad (2.62)$$

$$= - \begin{bmatrix} i\cos\phi & \gamma\sin\phi \\ -\gamma^{-1}\sin\phi & -i\cos\phi \end{bmatrix}^2 \quad (2.63)$$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ -3/i & 1 \end{bmatrix} \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \right\}^3 \quad (2.64)$$

$$= \left\{ \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3/l & 1 \end{bmatrix} \right\}^3 . \quad (2.65)$$

In these equations, γ , l , and ϕ are in general complex. It is interesting to note from Eq. (2.62) that, as opposed to scalars, matrices may have an infinite number of roots (and some matrices have no roots at all). Not all of the identity matrix factorizations above are given in terms of matrix primitives. However, in the case of Eq. (2.63), factorizations Eq. (2.51) and/or Eq. (2.52) may be substituted to accomplish this. These results lead to interesting optical systems such as the cat eye reflector [140] which, by design, has the same system matrix as a retroreflector expression (2.33).

Bilinear transformation

For every 2x2 matrix theory there exists an associated bilinear transformation. If a ratio parameter is defined

$$Z \equiv \frac{X}{Y} \quad (2.66)$$

then from Eq. (2.1) the corresponding transformation for the ratio parameter is

$$Z_2 = \frac{AZ_1 + B}{CZ_1 + D} . \quad (2.67)$$

This transformation is sometimes called the "ABCD law." In electrical theory Z is physically interpreted as an impedance, and in Gaussian beam theory it is related to the width and phase front radius of curvature of a Gaussian beam. In Jones calculus, it is interpreted as the "ellipse of polarization" [171]. The ratio parameter is interpreted as a reflection coefficient in the distributed feedback and fiber ring resonator theories.

Every system has a characteristic ratio parameter, Z_∞ , such that if Z_∞ is input to the system, the same Z_∞ is output. Thus by constraining the output ratio parameter to be equal to the input ratio parameter, it follows from Eq. (2.67) and the unimodularity condition, $AD - BC = 1$, that

$$A + B/Z_{\infty} = \frac{A + D}{2} \pm i \left[1 - \left(\frac{A + D}{2} \right)^2 \right]^{1/2} \quad (2.68)$$

$$= \exp[\pm i \theta] , \quad (2.69)$$

where

$$\cos \theta \equiv \frac{A + D}{2} . \quad (2.70)$$

It may be noted that Eq. (2.70) is identical to Eq. (2.37).

If a signal propagates through a system many times or through many identical systems, then Z may approach the value Z_{∞} . If Z approaches Z_{∞} the system is said to be "stable" with respect to Z , and this occurs when the complex magnitude [16]

$$|A + B/Z_{\infty}| > 1 , \quad (2.71)$$

where $|A + B/Z_{\infty}| = 1$ represents metastability, and $|A + B/Z_{\infty}| < 1$ represents instability. It may also be noted from Eq. (2.69) that

$$|A + B/Z_{\infty(+)}| |A + B/Z_{\infty(-)}| = 1 \quad (2.72)$$

so that if $Z_{\infty(+)}$ (the solution to Eq. (2.68) with the + sign) is stable then the other solution $Z_{\infty(-)}$ (the solution to Eq. (2.68) with the – sign) is unstable, and vice-versa.

INVARIANT TRANSFORMATIONS

It is often of interest that certain properties of a given system be conserved. For example, in the study of Gaussian beam modes in laser resonators, oscillation conditions require that X/Y be conserved. Lossless systems may be represented by a certain matrix form. This form may be obtained by applying the conservation of energy. A conserved quantity known as the Lagrange invariant may be used as an alternate method to system analysis.

Lagrange Invariant

The Lagrange invariant is a theorem that is often used in paraxial ray analysis. However, the theorem can be generalized so as to apply to any 2x2 transfer matrix theory. If two signals $[(X_a, Y_a)]$ and (X_b, Y_b) propagate through the same system, then

$$\begin{bmatrix} X_a \\ Y_a \end{bmatrix}_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_a \\ Y_a \end{bmatrix}_1 . \quad (2.73)$$

$$\begin{bmatrix} X_b \\ Y_b \end{bmatrix}_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_b \\ Y_b \end{bmatrix}_1 . \quad (2.74)$$

It can be shown by direct substitution that

$$(X_b Y_a - X_a Y_b)_2 = (AD - BC)(X_b Y_a - X_a Y_b)_1 . \quad (2.75)$$

In unimodular transfer matrix theories $AD - BC = 1$, and Eq. (2.75) may be written

$$X_b Y_a - X_a Y_b = \text{constant} . \quad (2.76)$$

This function $X_b Y_a - X_a Y_b$ is known as the Lagrange ray invariant when X represents ray position and Y represents ray slope (or angle).

Orthogonal Transformations

It may be desired to examine systems for which $X_2^2 + Y_2^2 = X_1^2 + Y_1^2$. This relation is valid for a class of "orthogonal" matrices. Orthogonal matrices are those in which $T_T = T^{-1}$, where the T subscript represents transposition (i.e. interchanging of the B and C elements). Orthogonal matrices have the properties that their determinants are unity in magnitude, and that the product of two orthogonal matrices is an orthogonal matrix. Similarly, the transpose and inverse of an orthogonal matrix are orthogonal matrices. Orthogonal matrices may generally be written

$$T_{\text{orthogonal}} = \begin{bmatrix} A & B \\ \mp B & \pm A \end{bmatrix} , \quad (2.77)$$

where $A^2 + B^2 = 1$. The rotation matrix [expression (2.30)] is an orthogonal matrix.

Unitary Transformations

It may be similarly desired to examine systems for which $X_2^* X_2 + Y_2^* Y_2 = X_1^* X_1 + Y_1^* Y_1$. This relation is valid for a class of "unitary" matrices. Unitary matrices are those in which $T_T^* = T^{-1}$, where the T subscript represents transposition (i.e. interchanging of the B and C elements) and the asterisk represents complex conjugation. Unitary matrices have the properties that the complex magnitude of their determinants is unity, and that the product of unitary matrices is a unitary matrix. Similar to orthogonal matrices, the transpose, conjugate, or inverse of a unitary matrix is unitary. Unitary matrices may generally be written

$$T_{\text{unitary}} = \begin{bmatrix} A & B \\ -B^* e^{i\delta} & A^* e^{i\delta} \end{bmatrix} \quad (2.78)$$

where δ is real and $A^* A + B^* B = 1$. The determinant of this matrix is $\exp(i\delta)$. When A and B are purely real and when $\delta = 0$ or $\delta = \pi$, then Eq. (2.78) reduces to Eq. (2.77), which is an orthogonal matrix.

ASSOCIATED MATRIX METHODS

For every 2x2 matrix method, there are several associated matrix methods. Some of these associated matrix methods may be equivalent to the original method, while others are generalizations of the original matrix method. In this section, several of these associated matrix methods are examined.

Staggered Linear Matrices

It is sometimes the case that neither X nor Y are physical observables, but that sums and differences of these quantities are. In this case, it is convenient to express a matrix method in terms of sums and differences of the signal vector quantities:

$$\begin{bmatrix} (X+\gamma Y)/2 \\ (X-\gamma Y)/2 \end{bmatrix}_2 = \begin{bmatrix} \frac{1}{2}(A+\gamma^{-1}B+\gamma C+D) & \frac{1}{2}(A-\gamma^{-1}B+\gamma C-D) \\ \frac{1}{2}(A+\gamma^{-1}B-\gamma C-D) & \frac{1}{2}(A-\gamma^{-1}B-\gamma C+D) \end{bmatrix} \begin{bmatrix} (X+\gamma Y)/2 \\ (X-\gamma Y)/2 \end{bmatrix}_1 \quad (2.79)$$

The determinant of this matrix is $AD - BC$. Here γ , which has units of X/Y , does not change. If γ varies, it may be of interest to define a matrix where only γ changes:

$$\begin{bmatrix} (X+\gamma Y)/2 \\ (X-\gamma Y)/2 \end{bmatrix}_2 = \begin{bmatrix} (1 + \gamma_1^{-1}\gamma_2)/2 & (1 - \gamma_1^{-1}\gamma_2)/2 \\ (1 - \gamma_1^{-1}\gamma_2)/2 & (1 + \gamma_1^{-1}\gamma_2)/2 \end{bmatrix} \begin{bmatrix} (X+\gamma Y)/2 \\ (X-\gamma Y)/2 \end{bmatrix}_1 . \quad (2.80)$$

The determinant of this matrix is γ_2/γ_1 .

Augmented Linear Matrices

Given the possibility of rotations, mirror reflections, and scaling in the XY plane, the next natural graphics operation that arises is translation [177]. However, simple translation in the XY plane cannot be performed with 2x2 matrix theories. An elegant way to account for translation is to augment the 2x2 matrix as a 3x3 matrix of the form

$$\begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} . \quad (2.81)$$

In this case the matrix for a simple translation is

$$T_{translation} = \begin{bmatrix} 1 & 0 & \chi_X \\ 0 & 1 & \chi_Y \\ 0 & 0 & 1 \end{bmatrix} \quad (2.82)$$

where χ_X and χ_Y are the amounts of translation in the X and Y axes respectively. In the paraxial ray theory, this translation matrix is interpreted physically as optical element or system misalignment [140]. Thus with the 3x3 theory, a lens, for example, is allowed to be displaced from the optic axis. A 3x3 electrical theory would allow ideal independent voltage and current sources distributed throughout the system. A 3x3 Jones calculus may include signal combining, and could account for multiple system inputs.

An important property of the matrix form Eq. (2.81) is that the E and F elements do not affect the A , B , C , and D elements in matrix multiplication. Furthermore, the determinant of the matrix is simply $AD - BC$, the same as the corresponding 2x2 matrix. An arbitrary augmented linear matrix may be factored in terms of translation matrices:

$$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & E \\ 0 & 1 & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.83)$$

$$= \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & (DE-BF)/(AD-BC) \\ 0 & 1 & (AF-CE)/(AD-BC) \\ 0 & 0 & 1 \end{bmatrix} \quad (2.84)$$

$$= \begin{bmatrix} 1 & 0 & E-(AF/C) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & F/C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.85)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & F-(DE/B) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & E/B \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.86)$$

Conjugate Linear Matrices

For systems represented by a complex matrix, a conjugate linear form may be of interest:

$$\begin{bmatrix} X \\ Y \\ X^* \\ Y^* \end{bmatrix}_2 = \begin{bmatrix} A & B & E & F \\ C & D & G & H \\ E^* & F^* & A^* & B^* \\ G^* & H^* & C^* & D^* \end{bmatrix} \begin{bmatrix} X \\ Y \\ X^* \\ Y^* \end{bmatrix}_1. \quad (2.87)$$

The determinant of the matrix is $|AD - BC|^2 + |EH - FG|^2 - |BG - DE|^2 - |AH - CF|^2 + 2\text{Re}/(EC - FG)(D^*F^* - B^*H^*)$. Expanding Gaussian beam matrices into a conjugate linear form allows one to include a phase conjugate mirror into the formalism:

$$\begin{bmatrix} u \\ Qu \\ (Qu)^* \end{bmatrix}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ Qu \\ (Qu)^* \end{bmatrix}_1. \quad (2.88)$$

This matrix has unity determinant.

Quadratic Matrices

Just as there are linear matrix methods, there are corresponding quadratic matrix methods:

$$\begin{bmatrix} X^2 \\ XY \\ Y^2 \end{bmatrix}_2 = \begin{bmatrix} A^2 & 2AB & B^2 \\ AC & AD+BC & BD \\ C^2 & 2CD & D^2 \end{bmatrix} \begin{bmatrix} X^2 \\ XY \\ Y^2 \end{bmatrix}_1. \quad (2.89)$$

The determinant of the matrix is $(AD - BC)^3$. It should be noted that information has been lost. As an example of this, it is impossible to find $\text{sgn}(X)$ given only X^2 .

Supplemented Quadratic Matrices

The previous squared matrices may be used to supplement the 2x2 matrix method as follows:

$$\begin{bmatrix} X \\ Y \\ X^2 \\ XY \\ Y^2 \end{bmatrix}_2 = \begin{bmatrix} A & B & 0 & 0 & 0 \\ C & D & 0 & 0 & 0 \\ 0 & 0 & A^2 & 2AB & B^2 \\ 0 & 0 & AC & AD+BC & BD \\ 0 & 0 & C^2 & 2CD & D^2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ X^2 \\ XY \\ Y^2 \end{bmatrix}_1. \quad (2.90)$$

The determinant of the matrix is $(AD - BC)^4$. As with Eq. (2.81), the signal vectors here may be augmented with unity and additional matrix elements may be added. The zeroes in the first two rows of the matrix in Eq. (2.90) insure that the matrix methods are self-consistent. However, in approximate analyses, they may be allowed to take on nonzero complex values. In addition to squared matrices, one may define cubic matrices, quartic matrices, quintic matrices, etc. Each of these matrix forms may be augmented, supplemented, staggered, and/or conjugated.

Complex Quadratic Matrices

Physically observable quantities are often represented as quadratic variables. This is the case in laser optics where irradiance, which is quadratic in the electric field, is the observable quantity. Similarly, charge density is a quadratic variable of the quantum mechanical wave function. In these and other cases, it is useful to consider matrices of the form

$$\begin{bmatrix} X^*X \\ Y^*Y \\ X^*Y \\ Y^*X \end{bmatrix}_2 = \begin{bmatrix} A^*A & B^*B & A^*B & B^*A \\ C^*C & D^*D & C^*D & D^*C \\ A^*C & B^*D & A^*D & B^*C \\ C^*A & D^*B & C^*B & D^*A \end{bmatrix} \begin{bmatrix} X^*X \\ Y^*Y \\ X^*Y \\ Y^*X \end{bmatrix}_1. \quad (2.91)$$

The first two items of the signal vector, X^*X and Y^*Y , are real, but the other two are complex. If X represents the complex voltage and Y represents the complex current in an electric circuit, then $\text{Re}(XY^*)$ is the complex power, which can be read directly off the matrix.

Staggered Complex Quadratic Matrices

The number of effective matrix elements in the complex quadratic matrices may be reduced if they are staggered. In particular, if X and Y have the same units, then we define

$$\begin{aligned} s_0 &= X^*X + Y^*Y \\ s_1 &= X^*X - Y^*Y \\ s_2 &= X^*Y + Y^*X \\ s_3 &= -i(X^*Y - Y^*X) \end{aligned} \quad (2.92)$$

If $X \equiv |X| \exp(i\theta_x)$, $Y \equiv |Y| \exp(i\theta_y)$, and $\theta \equiv \theta_y - \theta_x$, then the vectors may be rewritten as

$$\begin{aligned} s_0 &= |X|^2 + |Y|^2 \\ s_1 &= |X|^2 - |Y|^2 \\ s_2 &= 2|X||Y|\cos\theta \\ s_3 &= 2|X||Y|\sin\theta \end{aligned} \quad (2.93)$$

which have the property that $s_0^2 = s_1^2 + s_2^2 + s_3^2$. In terms of these variable, the staggered complex quadratic form is

$$\begin{bmatrix} s_o \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}_2 = \begin{bmatrix} \frac{1}{2}(A^*A + B^*B + C^*C + D^*D) & \frac{1}{2}(A^*A - B^*B + C^*C - D^*D) \\ \frac{1}{2}(A^*A + B^*B - C^*C - D^*D) & \frac{1}{2}(A^*A - B^*B - C^*C + D^*D) \\ \frac{1}{2}(A^*C + C^*A + B^*D + D^*B) & \frac{1}{2}(A^*C + C^*A - B^*D - D^*B) \\ -i/2(A^*C - C^*A + B^*D - D^*B) & -i/2(A^*C - C^*A - B^*D + D^*B) \end{bmatrix} \begin{bmatrix} s_o \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}_1 \quad (2.94)$$

When X and Y represent the Jones vector, then Eq. (2.94) is known as the Mueller matrix formalism.

SUMMARY

Transfer matrix methods represent powerful yet simple ways to perform optical analyses. In these methods, optical elements are represented by matrices, and system analysis consists merely of matrix multiplication.

There exists a large number of transfer matrix theories that are used to analyze different physical problems. Five of these matrix methods were briefly discussed in this chapter:

- 1) Paraxial ray matrices,
- 2) Gaussian beam matrices,
- 3) Jones calculus,
- 4) Reflection matrices,
- 5) Electric circuit matrices.

Ray matrices are used to analyze optical systems which change a light beam's position and slope. Gaussian beam matrices are used to analyze optical systems which change a light beam's spotsize and radius of phase curvature. Light beam polarization changes in an optical system may be traced with the use of Jones calculus. Optical systems which partially reflect an incident light beam may be analyzed with reflection matrices.

There were four matrix theorems discussed in this chapter:

- 1) Sylvester's theorem,
- 2) The reverse theorem,
- 3) Matrix factorization,
- 4) Bilinear transformation.

Sylvester's theorem is useful in the analysis and design of periodic optical systems. The reverse theorem is often used for multipass optical systems. Matrix factorization is used for system design, and the bilinear transformation provides additional insight into optical system behavior.

Three invariant transformations were examined in this chapter:

- 1) The Lagrange invariant,
- 2) Orthogonal transformations,
- 3) Unitary transformations.

The Lagrange invariant is used as alternate method of system analysis. In a phase plane representation of the state (X, Y) of an optical beam, orthogonal matrices represent constant-radius transformations ($X^2 + Y^2 = \text{constant}$). Similarly, optical systems represented by unitary matrices have $X^*X + Y^*Y = \text{constant}$. In the Jones calculus and the feedback matrices, lossless optical elements are represented by unitary matrices.

A variety of associated matrix methods were also discussed in this chapter.

CHAPTER III

GENERALIZED MATRIX THEOREMS

INTRODUCTION

For every 2x2 transfer matrix theory, there are several associated matrix methods, as described in the previous chapter. The most common of these associated matrix methods is the augmented matrix formulation. In addition to associated matrix methods, there are also several transfer matrix theorems. The purpose of this chapter is to explore two of these theorems, the reverse theorem and Sylvester's theorem, in greater detail, and to generalize them to the augmented 3x3 transfer matrix form.

REVERSE THEOREMS

Many optical systems contain some type of reflecting element which causes the light signal to propagate through all or part of an optical system backward. For example, standing wave and bidirectional ring laser oscillators contain optical signals which propagate through their intracavity optics in both directions. Reflective elements may also be used in optical system design where some desired effect is to be enhanced. This is the case in multipass amplifier schemes for increased amplification [178] or for distortion correction with phase conjugate mirrors. Similarly, multipass schemes may be used to decrease the transmission bandwidth of a filter. Another category of laser applications involves remote sensing and control [176], which may require reverse propagation through the optical system. Examples include remote sensing of the atmosphere, nondestructive evaluation, adaptive optics, fiber optic sensors, and microscopy. When the optical system is represented by a given matrix, then the corresponding matrix that represents

backward propagation through the system is of interest. This reverse matrix is also important if there are established system symmetry requirements or if there is a need for experimental determination of a system matrix.

Based on the examination of several types of optical elements and systems, one is sometimes able to divine the form of the reverse matrix. However, such a methodology ought not to be necessary, and systematic procedures are demonstrated to obtain the reverse matrix for a given matrix theory. The reverse matrix is often reported only for unimodular matrix theories. However, many matrix theories are unimodular only for some special case. For example, Jones calculus is unimodular only when the optical system is lossless and when absolute phase is ignored. A notable extension of the Jones calculus accounts, to first order, for polarization dependent Fresnel reflection and refraction for nonnormal incidence. This "extended Jones matrix method" [179] retains the simple 2×2 form, but is inherently nonunimodular. Of course, if the birefringent optical system contains polarizers then the system is represented as a zero determinant matrix, and there is no unique reverse matrix. The characteristic matrix method for light propagation in stratified media [180] is unimodular only when the media and boundaries are lossless [181]. Similar restrictions are involved in transfer matrices used for distributed feedback structures [182] and fiber ring resonators [183]. In the case of Gaussian beams and paraxial rays, the unimodularity condition occurs only when the medium at the output has the same refractive properties as the medium at the input [59]. However, the nonunimodular reverse matrix has been obtained for the Gaussian beam matrix method [134]. The generalization of the nonunimodular reverse matrix concept to other matrix theories is addressed in this chapter.

For every 2×2 matrix method there is an augmented matrix which corresponds to a 3×3 matrix method. The form of the 3×3 matrix of interest here is much simpler than the general 3×3 matrix. In both the paraxial ray matrix theory and in the Gaussian beam

theory, the 3×3 matrix method allows the designer to trace paraxial light rays and Gaussian beams through misaligned optical systems [140]. This 3×3 formalism may be applied to, for example, the design of pulse compressors [184] - [186]. Similarly, 3×3 matrix methods are necessary to study electrical circuits that contain intranetwork independent voltage and current sources. Another example exists in computer graphics where operations are performed on subfigures in a picture by means of 2×2 matrix multiplication. However, to perform translation, an augmented 3×3 matrix description is necessary [177].

Universal Reverse Matrix Properties

The purpose of this section is to demonstrate a systematic procedure to obtain the reverse matrix for a given matrix theory. The process does not require the usual inspection of individual system elements [134] [161] [187]. The secondary purpose of this section is to use these systematic procedures to obtain new reverse matrices for several matrix theories. In particular, the reverse matrix for 3×3 electric circuit matrices is found here. These results may be used to study reverse propagation through electric circuits with intranetwork independent current and voltage sources. In the Gaussian beam theory, the 2×2 reverse matrices governing the beam's spotsize and phase front curvature are known [187]. The reverse matrix for nonunimodular 3×3 paraxial ray matrices is also found. Previously, only the nonunimodular 2×2 form [134] and the unimodular 3×3 form [188] have been discussed. The reverse matrix for the Jones Calculus matrix method has been reported for 2×2 matrices [161] [176]. Here, the nonunimodular 3×3 reverse matrix is derived. This generalization may account for multiple-input optical systems with loss or gain. The nonunimodular 3×3 reverse matrix is also found for the matrix theories governing distributed feedback lasers and waveguides and fiber ring resonators.

The reverse theorems do not apply to optical systems represented by a zero deter-

minant matrix. In these systems there may be several different inputs that yield the same output. Though there are different reverse matrices for different matrix methods, there are certain universal properties that these reverse matrices all share. For example, the reverse of a product of matrices is the product of the reverse of each of these matrices in reverse order. In equation form, this may be written

$$(T_1 T_2 \cdots T_s)_R = T_{sR} \cdots T_{2R} T_{1R} . \quad (3.1)$$

The justification of Eq. (3.1) is suggested by Fig 3-1. In that figure, the matrix T , governing propagation from left to right is defined as a product of submatrices: $T_1 T_2 T_3 T_4$. Similarly, new matrices going from right to left may be defined. However, it is evident from the figure that these matrices correspond to the reverse matrices T_R , T_{4R} , T_{3R} , T_{2R} , and T_{1R} . It follows that, to obtain T_R , the reverse submatrices must be multiplied in reverse order. This conclusion is independent of whether the matrix is 2x2 or of higher order, and is independent of the matrix theory being studied. Similarly, there is no assumption about the determinant except that it is nonzero to assure the existence of the inverse. For Gaussian beam theory, this property of the unimodular 2x2 reverse matrix has been discussed previously [187].

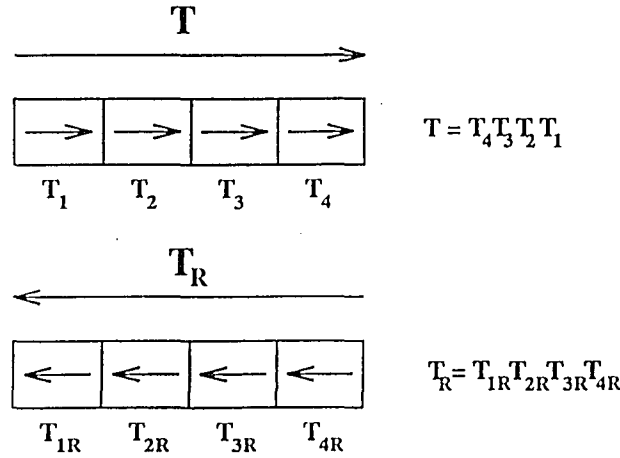


Figure 3-1 Schematic demonstration that the reverse of a product of matrices is the product of reverse matrices in reverse order independent of matrix theory, i.e. $(T_1 T_2 T_3 T_4)_R = T_{4R} T_{3R} T_{2R} T_{1R}$.

As a special case of this theorem, suppose each of the submatrices $T_1 - T_s$ are identical, then it follows that

$$(T^s)_R = (T_R)^s . \quad (3.2)$$

This is the intuitive result that propagation through s identical systems backward is the same as propagation backward through a single system s times. Though it was assumed that the exponent s in Eq. (3.2) is a positive integer, it is valid for any exponent s .

Another special case of Eq. (3.1) is

$$T_R T = (T_R T)_R . \quad (3.3)$$

The matrix $T_R T$ represents forward propagation through a system followed by reverse propagation through that same system. In standing-wave laser theory, this matrix often corresponds to a "round-trip". Now that some general properties of reverse matrices have been described, the specific reverse matrices for several matrix theories will be discussed.

GENERALIZED REVERSE MATRICES

The purpose of this section is to demonstrate a simple method whereby reverse matrices may be derived for a large variety of transfer matrix methods. With this reverse

matrix derivation method, it is straightforward to obtain reverse matrices for augmented 3x3 matrix methods.

Paraxial Ray Matrices

The purpose of this subsection is to derive the form of the reverse matrix that applies to paraxial light rays. At some position τ along the optic axis, the X signal vector component represents the position of a light ray, and the Y component represents the slope of the light ray. For this derivation it is important to note that $Y = dX/d\tau$.

The reverse matrix governs the propagation of the signal going backward and starting from the "output". Therefore the definition of T_R is

$$T_R \equiv \text{Input} \mid_{\text{going backward}} \mid \text{given} \quad \text{Output} \mid_{\text{going backward}} . \quad (3.4)$$

As before, the propagation of X and $dX/d\tau$ are governed by the transformation matrix

$$\begin{bmatrix} X \\ \frac{\partial X}{\partial \tau} \end{bmatrix}_2 = T \begin{bmatrix} X \\ \frac{\partial X}{\partial \tau} \end{bmatrix}_1 , \quad (3.5)$$

where T is the system matrix. Multiplying both sides of Eq. (3.5) by T^{-1} yields the formula for the input ray position and slope given the output ray position and slope:

$$\begin{bmatrix} X \\ \frac{\partial X}{\partial \tau} \end{bmatrix}_1 = T^{-1} \begin{bmatrix} X \\ \frac{\partial X}{\partial \tau} \end{bmatrix}_2 . \quad (3.6)$$

The direction of the signal is $+\tau$. If the signal is propagating in the reverse direction, then τ is replaced with $-\tau$. Thus, Eq. (3.6) may be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ \frac{\partial X}{\partial(-\tau)} \end{bmatrix}_1 = T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ \frac{\partial X}{\partial(-\tau)} \end{bmatrix}_2 . \quad (3.7)$$

Multiplying both sides by the matrix in Eq. (2.31), and noticing that

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.8)$$

reduces Eq. (3.7) to

$$\begin{bmatrix} X \\ \frac{\partial X}{\partial(-\tau)} \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ \frac{\partial X}{\partial(-\tau)} \end{bmatrix}_2. \quad (3.9)$$

However, from the definition of the reverse matrix Eq. (3.4), it follows that the reverse matrix is

$$T_R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3.10)$$

Since $(T^{-1})^2 = (T^2)^{-1}$, it can be readily seen from Eq. (3.10) that $(T_R)^2 = (T^2)_R$. By induction, a general property of reverse matrices Eq. (3.2) follows. Similarly, Eq. (3.1) can be seen. Thus, this specific reverse matrix has the same property that was shown to be a general property of reverse matrices.

The specific form of T^{-1} is well known, and Eq. (3.10) can be reduced to

$$T_R = \frac{1}{AD-BC} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{AD-BC} \begin{bmatrix} D & B \\ C & A \end{bmatrix}. \quad (3.11)$$

In the special case that the system matrix is unimodular, this is the known result. The nonunimodular reverse matrix has been recently derived, and its validity has been checked for several optical elements [134]. This result, along with the 2x2 matrix results from the previous section, is listed in Table I. The symmetry condition is attained if a matrix is equal to its reverse matrix. If the matrix is unimodular, then $AD-BC=1$, and the system is symmetric if $A = D$.

TABLE I
REVERSE MATRICES FOR SEVERAL MATRIX THEORIES

Transfer Matrix Theory	2x2 Reverse Matrix	3x3 Reverse Matrix
Paraxial Ray Matrices Gaussian Beam Matrices Electric Circuit Matrices Wronskian-type Matrices	$\frac{1}{AD-BC} \begin{bmatrix} D & B \\ C & A \end{bmatrix}$	$\frac{1}{AD-BC} \begin{bmatrix} D & B & BF-DE \\ C & A & -(CE-AF) \\ 0 & 0 & AD-BC \end{bmatrix}$
Jones Calculus	$\frac{1}{AD-BC} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}^*$	$\frac{1}{AD-BC} \begin{bmatrix} D & -B & BF-DE \\ -C & A & CE-AF \\ 0 & 0 & AD-BC \end{bmatrix}^*$
Feedback Matrices Fiber Ring Matrices	$\frac{1}{AD-BC} \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}$	$\frac{1}{AD-BC} \begin{bmatrix} A & -B & CE-AF \\ -C & D & BF-DE \\ 0 & 0 & AD-BC \end{bmatrix}$

The use of 3x3 matrices to account for misaligned optical systems has become popular, and the 3x3 reverse matrix for unimodular ray optical systems is known [188]. However, the nonunimodular 3x3 reverse matrix has not been previously reported. From the developments here, it is clear that retracing the steps in Eqs. (3.4)-(3.9) for systems of the augmented form given in Eq. (2.81), results in

$$T_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.12)$$

$$= \frac{1}{AD-BC} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D & -B & BF-DE \\ -C & A & CE-AF \\ 0 & 0 & AD-BC \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.13)$$

$$= \frac{1}{AD-BC} \begin{bmatrix} D & B & BF-DE \\ C & A & AF-CE \\ 0 & 0 & AD-BC \end{bmatrix}. \quad (3.14)$$

This result, along with the 3x3 matrix results from the previous section, is also listed in Table I.

None of the properties unique to paraxial ray matrices were used in the derivation. Besides the postulating of the existence of an inverse, the only assumption was that the Y component of the signal vector was the τ derivative of the X component of the signal vector. Thus this result is not unique to ray matrices. These reverse matrices also apply to the Gaussian beam matrix formalism, the electric circuit matrix theory, and other Wronskian type matrix theories where $Y=dX/d\tau$.

Jones Calculus

The reverse matrix has been found for the ray matrix formalism, the Gaussian beam matrix formalism, and for the two-port electric circuit theory. The reverse matrix takes on the same form for each of these. However, for the Jones Polarization Calculus, the reverse matrix is different, and must be calculated separately.

Rather than derive the 2x2 and then the 3x3 reverse matrix, the more general 3x3 case is derived. The Jones matrix is allowed to have the form

$$T = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.15)$$

The 2x2 reverse matrix becomes a simplified special case where $E = F = 0$. The transformation of the augmented Jones vectors is

$$\begin{bmatrix} A_x \exp[i\phi_x] \\ A_y \exp[i\phi_y] \\ 1 \end{bmatrix}_2 = T \begin{bmatrix} A_x \exp[i\phi_x] \\ A_y \exp[i\phi_y] \\ 1 \end{bmatrix}_1, \quad (3.16)$$

where A_x , A_y , ϕ_x , and ϕ_y are real. Proceeding as in the previous reverse matrix derivation, both sides of Eq. (3.16) are premultiplied by T^{-1} :

$$\begin{bmatrix} A_x \exp[i\phi_x] \\ A_y \exp[i\phi_y] \\ 1 \end{bmatrix}_1 = T^{-1} \begin{bmatrix} A_x \exp[i\phi_x] \\ A_y \exp[i\phi_y] \\ 1 \end{bmatrix}_2. \quad (3.17)$$

When the optical system is lossless, the matrices are unitary, and reversal of the Jones vectors implies reversal of phase. Thus,

$$\begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \end{bmatrix}_1^* = T^{-1} \begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \end{bmatrix}_2^*, \quad (3.18)$$

where the asterisk represents complex conjugation. Taking the complex conjugate of both sides yields

$$\begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \end{bmatrix}_1 = (T^{-1})^* \begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \end{bmatrix}_2. \quad (3.19)$$

Thus, from Eq. (3.4) it follows that the reverse matrix is

$$T_R = (T^{-1})^*. \quad (3.20)$$

When the matrix is unitary, this result becomes the same as Eq. (2.39). However, this result applies to lossless systems represented by unitary 3x3 matrices. Care should be taken in the case of Faraday rotators and media with optical activity, however, since though they have the same forward matrix, their reverse matrices may differ. Equation (3.20) is listed in Table I. This result has been previously reported only for 2x2 matrices [176]. If the matrix is unimodular, then the corresponding optical system is symmetric if B and C are pure imaginary and if $A = D^*$. For these Jones matrices, the reverse matrix has been defined so that the transverse axes are unchanged. Thus the reverse coordinate system is left-handed. When this is undesirable, one may, for example, change the sign of A_y in Eq. (3.18). As a final note, it is interesting that the transpose may be written for nonzero determinant matrices as

$$T_T = (AD - BC) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (3.21)$$

Feedback Matrices

As a further example of the methodology for finding the reverse matrix, the reverse matrix is found for the feedback matrix theory. Here the electric field is separated into right and left going waves which form the signal vector. For generality, it is postulated

that matrices take the more general form Eq. (3.15). Thus the signal vector is augmented with unity as

$$\begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_2 = T \begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_1. \quad (3.22)$$

Proceeding as in the previous subsections, both sides are multiplied by T^{-1} :

$$\begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_1 = T^{-1} \begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_2. \quad (3.23)$$

The signal vector can again be written as a matrix multiplied by the corresponding signal vector traveling in the opposite direction:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_1 = T^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_2. \quad (3.24)$$

Multiplying both sides of Eq. (3.24) on the left by the appropriate matrix yields

$$\begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_2. \quad (3.25)$$

Thus it follows from Eq. (3.4) that

$$T_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.26)$$

The result of this calculation is included in Table I. In the 2x2 special case, Eq. (3.26) reduces to

$$T_R = \frac{1}{AD-BC} \begin{bmatrix} A & -C \\ -B & D \end{bmatrix}. \quad (3.27)$$

If $AD-BC=1$, then the system represented by the matrix T is symmetric if $B = -C$.

SYLVESTER'S THEOREM

Ubiquitous 2x2 transfer matrix methods are commonly used to study a wide variety of problems in optics [131][140] and in other areas of engineering and physics. With

these methods system analysis involves only 2×2 matrix multiplication. When the optical system under consideration has properties that vary periodically, the system matrix must be multiplied by itself several times. The mathematical formula governing this procedure is known as Sylvester's theorem. Typically the somewhat obscure general form of Sylvester's theorem is only reported in mathematical texts, or else one encounters the simplest explicit special case for unimodular 2×2 matrices. However, it has long been desired to obtain a practical form of Sylvester's theorem for 6-element 3×3 matrices [148]. One purpose of this chapter is to develop Sylvester's theorem for such matrices having arbitrary determinants.

Periodic systems occur naturally as in, for example, homogeneous crystals which contain periodic crystalline planes. Similarly, periodic sequences of lenses or apertures may be used for optical waveguiding [22] [189]. In the limit of small period, these "lens-waveguides" have the same propagation characteristics as inhomogeneous lenslike media. Conventional lasers, Fabry-Perot interferometers, optical delay lines [190], and multipass resonators [58] represent important classes of periodic optical systems which may have the same mode structure as these lens-waveguides. Periodically perturbed optical fibers are considered to be a possible consequence of defective manufacturing techniques or distortions in multifiber cables [81]. Acoustic waves are sometimes used to generate a periodic refractive profile in acousto-optic media. An important class of particle accelerators is periodic [191] [192]. In addition to applications with these systems, Sylvester's theorem may also be used in the analysis and design of fan and folded Solc filters, distributed feedback waveguides and lasers, twisted nematic liquid crystals, Bragg filters, and surface wave devices [193].

Sylvester's theorem is usually only reported for the special case of unimodular matrix theories. However, many matrix theories are unimodular only when the optical system is subject to some restrictive constraints. For example, Jones calculus [161] -

[168] is unimodular only when the optical system is lossless and when absolute phase is ignored. This situation is certainly invalid when the optical system contains polarizing elements, or when polarization dependent reflection and refraction for nonnormal incidence is accounted for [194]. Only the assumption of losslessness allows other matrix methods such as those involving characteristic matrices for stratified media [195], transfer matrices for distributed feedback structures [196], and transfer matrices for fiber ring resonators [197] to be unimodular as well.

For many 2x2 transfer matrix methods there are corresponding augmented 3x3 matrix methods [131]. The form of the 3x3 matrix of interest here, however, contains only six independent elements. In both the paraxial ray matrix theory and in the Gaussian beam theory, the 3x3 matrix method allows the designer to trace paraxial light rays and Gaussian beams through misaligned optical systems [140] [148]. This 3x3 formalism may also be applied to, for example, the design of pulse compressors [184], dispersive laser cavities [185] [186], and particle accelerators [198]. Similarly, 3x3 matrix methods are necessary to study two-port electrical circuits that contain intranetwork independent voltage and current sources and for computer graphics manipulations in which object translation is required [177].

Sylvester's Matrix Polynomial Theorem for $n \times n$ Matrices

In transfer matrix applications, system analysis involves simple matrix multiplication. When the system under consideration is periodic, the system matrix must be multiplied by itself several times and the formula for the s th power of an $n \times n$ matrix is of interest. This formula, named after the 19th century mathematician James Joseph Sylvester (1814-1897), is Sylvester's matrix polynomial theorem reported in 1882 [175], and sometimes referred to as "Sylvester's theorem" [199], "Sylvester's formula" [200], or the "Lagrange-Sylvester formula" [201]. Sylvester, a pioneer in linear algebra, is also

responsible for the term "matrix", his usage of which began in 1850. There are differing opinions on the name to be given to the 2x2 special case of Sylvester's matrix polynomial theorem. The reason for this may stem in part from the fact that this important case of the theorem was reported in 1858 by Sylvester's close friend Arthur Cayley (1821-1895). In fact, Cayley, the creator of modern matrix theory, had written the 2x2 theorem in his original paper on matrices [174]. In a standard optics text, Born and Wolf report the theorem but do not give it a name [202] and others have done the same [203] [204]. Perhaps because the 2x2 theorem can be written in terms of Chebyshev (Tchebyscheff) polynomials, it has recently been called "Chebyshev's identity" [205], and others have followed in this usage [193] [196]. The most common and more appropriate name used, however, is "Sylvester's theorem" [131] [140] [144] [146] [148] [206]; and this convention is followed here. Specifically, we will mean by "Sylvester's theorem" the 2x2 unimodular (unit determinant) special case of the more general "Sylvester's matrix polynomial theorem". (It may be noted that J. J. Sylvester is responsible for many other theorems, several of which are sometimes also referred to nondescriptively as "Sylvester's theorem.")

Many of the results of this study are derivable directly from Sylvester's matrix polynomial theorem. While this general theorem is known in the mathematics literature, its relevance in optics is not so well known. As an example of its use, Sylvester's theorem (2x2 unimodular case) is derived from it in this section. The procedure to evaluate Sylvester's theorem in the most general case is also discussed. This process, though straightforward in principle, is sometimes almost intractable in practice. Thus several special cases of Sylvester's theorem are identified in this section.

Suppose T is an $n \times n$ matrix, I is the $n \times n$ identity matrix, and λ is a scalar. A "characteristic" equation may be defined in terms of a determinant:

$$p(\lambda) \equiv |T - \lambda I| = 0 . \quad (3.28)$$

The scalar roots of this characteristic equation, $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix T . If $p(\lambda)$ has no multiple roots, the s th power of the matrix T is

$$T^s = \sum_{j=1}^n \left[\lambda_j^s \prod_{i \neq j} \frac{\lambda_i I - T}{\lambda_i - \lambda_j} \right]. \quad (3.29)$$

This is Sylvester's matrix polynomial theorem. When the eigenvalue Eq. (3.28) has non-distinct roots, there is a more general confluent form of the theorem [207].

Sylvester's Matrix Polynomial Theorem for 2x2 Matrices

To illustrate the usage of Eq. (3.29), the simplest nontrivial matrix exponent is considered. For this 2x2 matrix case $n = 2$ and Eq. (3.29) becomes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \sum_{j=1}^2 \left\{ \lambda_j^s \prod_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \begin{bmatrix} \lambda_i - A & -B \\ -C & \lambda_i - D \end{bmatrix} \right\}, \quad (3.30)$$

$$= \frac{\lambda_1^s}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 - A & -B \\ -C & \lambda_2 - D \end{bmatrix} + \frac{\lambda_2^s}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 - A & -B \\ -C & \lambda_1 - D \end{bmatrix}, \quad (3.31)$$

$$= \begin{bmatrix} A \frac{\lambda_2^s - \lambda_1^s}{\lambda_2 - \lambda_1} + \lambda_1 \lambda_2 \frac{\lambda_1^{s-1} - \lambda_2^{s-1}}{\lambda_2 - \lambda_1} & B \frac{\lambda_2^s - \lambda_1^s}{\lambda_2 - \lambda_1} \\ C \frac{\lambda_2^s - \lambda_1^s}{\lambda_2 - \lambda_1} & D \frac{\lambda_2^s - \lambda_1^s}{\lambda_2 - \lambda_1} + \lambda_1 \lambda_2 \frac{\lambda_1^{s-1} - \lambda_2^{s-1}}{\lambda_2 - \lambda_1} \end{bmatrix}. \quad (3.32)$$

Given the two eigenvalues of a matrix T , Eq. (3.32) is the formula for T^s . The eigenvalues, as mentioned above, can be obtained from the characteristic Eq. (3.28), which for a 2x2 matrix is

$$p(\lambda) = (A - \lambda)(D - \lambda) - BC = 0. \quad (3.33)$$

This equation may be rewritten in standard quadratic equation form:

$$\lambda^2 - (A + D)\lambda + (AD - BC) = 0. \quad (3.34)$$

There are several ways to proceed. It is clear that the trace $A + D$, and the determinant $AD - BC$ are important quantities, and the eigenvalues depend directly on these quantities. For this derivation only, the special case where the determinant is unity is

considered. Indeed, this special case applies to many matrix theories including ray matrices, Gaussian beam matrices, two-port electric circuit matrices, and the normalized form of the matrices of Jones calculus. The trace is allowed to be arbitrary. If the definition

$$\cos\theta \equiv \frac{A+D}{2} \quad (3.35)$$

is introduced, then the eigenvalues from the quadratic characteristic Eq. (3.34) are

$$\lambda_{1,2} = \exp[\pm i\theta] , \quad (3.36)$$

and Eq. (3.32) reduces to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \frac{1}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \sin[(s-1)\theta] & B \sin(s\theta) \\ C \sin(s\theta) & D \sin(s\theta) - \sin[(s-1)\theta] \end{bmatrix} , \quad (3.37)$$

which is the standard form of Sylvester's theorem.

Alternate Forms and Evaluation of Sylvester's Theorem

To emphasize the polynomial nature of the solution, Eq. (3.37) is sometimes written in terms of Chebyshev polynomials of the second kind [202]:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \begin{bmatrix} AU_{s-1}(x) - U_{s-2}(x) & BU_{s-1}(x) \\ CU_{s-1}(x) & DU_{s-1}(x) - U_{s-2}(x) \end{bmatrix} , \quad (3.38)$$

where

$$x \equiv (A+D)/2 \quad (3.39)$$

and

$$U_s(x) = \frac{\sin[(s+1)\cos^{-1}(x)]}{(1-x^2)^{1/2}} . \quad (3.40)$$

The first several Chebyshev polynomials are

$$\begin{aligned}
U_0(x) &= 1 , \\
U_1(x) &= 2x , \\
U_2(x) &= 4x^2 - 1 , \\
U_3(x) &= 8x^3 - 4x , \\
U_4(x) &= 16x^4 - 12x^2 + 1 , \\
U_5(x) &= 32x^5 - 32x^3 + 6x , \\
U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1 .
\end{aligned} \tag{3.41}$$

Additional polynomials may be obtained by the recursion relation [208]

$$U_{n+1} - 2xU_n + U_{n-1} = 0 . \tag{3.42}$$

The unimodular 2x2 form of Sylvester's theorem has been derived from Sylvester's matrix polynomial theorem. Alternately, one may derive it from first principles, and this derivation is given in appendix A of Ref. 133. A similar derivation is also given in Ref. 140. Conceptually simpler is the inductive proof [206] which is reported in appendix B of Ref. 133. These two appendices have been used to explore the range of validity of Sylvester's theorem. For example, these appendices suggest that Sylvester's theorem may also apply to roots of matrices. In appendix C of Ref. 133, roots are specifically considered, and it is shown that Sylvester's theorem is also valid for these matrix roots [174] [199]. The fact that Sylvester's theorem applies to integer powers and roots of matrices is suggestive that it applies to arbitrary rational powers. The proof of this is given in appendix D of Ref. 133. The proofs in these appendices are crucial for a rigorous understanding of Sylvester's theorem and its applications to matrix optics. With the proofs given in Ref. 133, this section contains a comprehensive summary of the properties of this important theorem.

In general, each of the $ABCD$ matrix elements in Sylvester's theorem may be complex. The procedure for evaluating the matrix Eq. (3.37) is begun by calculating θ . This "angle" may be determined explicitly by combining the Euler relation with Eq. (3.35) to yield

$$\theta = -i \ln \left\{ \frac{A+D}{2} \pm \left[\left(\frac{A+D}{2} \right)^2 - 1 \right]^{1/2} \right\} . \quad (3.43)$$

The complex square root in this formula may be evaluated using

$$(a + ib)^{1/2} = \pm \left\{ \left[\frac{(a^2 + b^2)^{1/2} + a}{2} \right]^{1/2} + i \operatorname{sgn}(b) \left[\frac{(a^2 + b^2)^{1/2} - a}{2} \right]^{1/2} \right\} , \quad (3.44)$$

where the signum function $\operatorname{sgn}(b) \equiv b/|b|$ has been used. The complex natural logarithm may be separated into real and imaginary parts using the relationship

$$\ln(a + ib) = \ln[(a^2 + b^2)^{1/2}] + i [\tan^{-1}(b/a)] . \quad (3.45)$$

It should be noted that this complex logarithm is in general multivalued since the arctangent function in Eq. (3.45) has an infinite number of branches. Finally, the elements of the matrix Eq. (3.37) may be evaluated by noting that

$$\sin(a + ib) = \sin(a) \cosh(b) + i \cos(a) \sinh(b) . \quad (3.46)$$

This procedure for evaluating Eq. (3.37) simplifies if $(A+D)/2$ is real. However, if $(A+D)/2$ is greater than unity in magnitude, then from Eq. (3.43), θ is purely imaginary. In this case it is convenient to rewrite Sylvester's theorem in another form. If $\theta \equiv i\phi$, then Eq. (3.37) may be written

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \frac{1}{\sinh \phi} \begin{bmatrix} A \sinh(s\phi) - \sinh[(s-1)\phi] & B \sinh(s\phi) \\ C \sinh(s\phi) & D \sinh(s\phi) - \sinh[(s-1)\phi] \end{bmatrix} , \quad (3.47)$$

where

$$\cosh \phi = \frac{A+D}{2} . \quad (3.48)$$

Special Cases of Sylvester's Theorem

Due to the somewhat complicated form of Sylvester's theorem, it is useful to consider special cases of Eq. (3.37) where the matrix elements taken on specific forms. One such special case exists when $A = D \equiv \cos \theta$, and $\chi \equiv (-B/C)^{1/2}$. Here, Sylvester's Theorem reduces to

$$\begin{bmatrix} \cos\theta & \chi\sin\theta \\ -\chi^{-1}\sin\theta & \cos\theta \end{bmatrix}^s = \begin{bmatrix} \cos(s\theta) & \chi\sin(s\theta) \\ -\chi^{-1}\sin(s\theta) & \cos(s\theta) \end{bmatrix}. \quad (3.49)$$

In the limit as θ approaches zero, this equation has two different forms of interest:

$$\begin{bmatrix} 1 & \chi_1 \\ 0 & 1 \end{bmatrix}^s = \begin{bmatrix} 1 & s\chi_1 \\ 0 & 1 \end{bmatrix}, \quad (3.50)$$

$$\begin{bmatrix} 1 & 0 \\ \chi_2 & 1 \end{bmatrix}^s = \begin{bmatrix} 1 & 0 \\ s\chi_2 & 1 \end{bmatrix}. \quad (3.51)$$

For nonunimodular matrices, an important special case of Sylvester's matrix polynomial theorem is

$$\begin{bmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{bmatrix}^s = \begin{bmatrix} \chi_1^s & 0 \\ 0 & \chi_2^s \end{bmatrix}. \quad (3.52)$$

While this result is almost obvious, it may be derived by solving the eigenvalue Eq. (3.34) directly, and using Eq. (3.32). The eigenvalues in this case are A and D .

If s is constrained to be an integer, then Sylvester's matrix polynomial theorem may be used to evaluate the interesting off-diagonal matrix form

$$\begin{bmatrix} 0 & \chi_1 \\ \chi_2 & 0 \end{bmatrix}^{2s} = (\chi_1\chi_2)^s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (3.53)$$

$$\begin{bmatrix} 0 & \chi_1 \\ \chi_2 & 0 \end{bmatrix}^{2s+1} = (\chi_1\chi_2)^s \begin{bmatrix} 0 & \chi_1 \\ \chi_2 & 0 \end{bmatrix}. \quad (3.54)$$

These results may also be found from Eqs. (3.32) and (3.34). The eigenvalues are $\pm(BC)^{1/2}$. When $\chi_1\chi_2 = -1$, these results follow from Eq. (3.49) with $\theta = \pi/2$. These last two matrix Eqs. (3.53) and (3.54) are of special interest in the analysis of confocal resonators. In this case, this off-diagonal matrix represents the transformation of a Gaussian beam after propagating from the center of the resonator back to the center of the resonator.

In the synthesis of both multipass and periodic systems, factorizations and roots of matrices are often of interest. For example, it is sometimes desired that an optical system

be transparent in the sense that the system matrix is made to be the identity matrix. Some factorizations of the identity matrix were given in chapter II. The identity matrix taken to any positive integer power is the identity matrix, and this result may be obtained from either Eqs. (3.50) or (3.51). However, it is clear from Eq. (2.62) that the identity matrix has an infinite number of distinct square roots (and cube roots too). These do not follow from Eqs. (3.50) or (3.51) since it was assumed that the eigenvalues in Eq. (3.36) were distinct. The difficulty may be readily seen from appendix A of Ref. 133. Since the eigenvalues are equal, the eigenvalue matrix Λ commutes with the matrix M . In this case, the substitution $T = M^{-1}\Lambda M$ reduces to $T = \Lambda$ which is not useful, and an alternate technique must be employed. In particular, roots must be examined specifically,

$$T^s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad (3.55)$$

and Sylvester's theorem is well-suited to this problem. The unimodular form of Sylvester's theorem may be used to reduce this equation to four scalar equations with four unknowns:

$$\frac{A \sin(s\theta) - \sin[(s-1)\theta]}{\sin\theta} = 1 , \quad (3.56)$$

$$\frac{C \sin(s\theta)}{\sin\theta} = 0 , \quad (3.57)$$

$$\frac{B \sin(s\theta)}{\sin\theta} = 0 , \quad (3.58)$$

$$\frac{D \sin(s\theta) - \sin[(s-1)\theta]}{\sin\theta} = 1 , \quad (3.59)$$

where θ is defined in Eq. (3.43). The trivial solutions are $A = D = \pm 1, B = C = 0$. Alternately, Eqs. (3.57) and (3.58) are satisfied if

$$\sin(s\theta) = 0 , \quad (3.60)$$

$$\sin\theta \neq 0 . \quad (3.61)$$

After the use of a trigonometric identity, it may be seen that the other two equations are

solved when

$$\cos(s\theta) = 1 \quad . \quad (3.62)$$

Equations (3.60) and (3.62) are satisfied when

$$s\theta = 2k\pi \quad , \quad (3.63)$$

where k may be any integer. To avoid duplicating solutions, the restriction

$$0 \leq k \leq s/2 \quad (3.64)$$

is made [58]. Equation (3.61) may be combined with Eq. (3.63) to obtain a restriction on s , and it follows that $s \neq 2$. Thus, the identity matrix has no nontrivial unimodular square roots.

However, there are nontrivial nonunimodular square roots, those with a determinant of -1. As may be seen in the next section, the third criteria Eq. (3.62) becomes

$$\cos(s\theta) = (AD - BC)^{-s/2} \quad . \quad (3.65)$$

If $s = 2$ and the determinant is -1, then the three criteria Eqs. (3.60), (3.61), and (3.65) are satisfied when $\theta = \pi/2$. The nonunimodular relationship

$$\cos\theta = \frac{1}{2}(A + D)(AD - BC)^{-1/2} \quad (3.66)$$

may be used, and it follows that for this value of θ , $D = -A$. Since the determinant is -1, the result may be written

$$I^{1/2} = \begin{bmatrix} \cos\phi & \gamma\sin\phi \\ \gamma^{-1}\sin\phi & -\cos\phi \end{bmatrix} \quad , \quad (3.67)$$

which was reported, but not derived, in chapter 2. As an additional result, if a matrix T is a known s th root of the identity matrix, then $I^{1/2}TI^{1/2}$ is also an s th root of the identity matrix. Since a nonunimodular Sylvester's theorem has been used, these results provide an additional motivating factor for the derivation of generalized Sylvester's theorems. These generalized theorems are considered in the next section.

GENERALIZED SYLVESTER THEOREMS

It has long been desired to obtain Sylvester's theorem for 6-element 3x3 matrices [148]. The purpose of this section is to derive nonunimodular 2x2 and 3x3 forms of Sylvester's theorem. The methodology used is discussed, and it is straightforward to extend the results here to other higher order matrices.

Nonunimodular 2x2 Sylvester Theorems

There is more than one method that may be used to find the nonunimodular form of Sylvester's theorem. From Sylvester's matrix polynomial theorem, it is clear that the desired matrix may be obtained from Eq. (3.32) with the eigenvalues derived from the general solution to the characteristic Eq. (3.34). One may also derive it directly in the same manner as is done in appendix A of Ref. 133. Alternately, if the determinant, $\tau = AD - BC$, is nonzero, it may be factored out, and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \tau^{s/2} \begin{bmatrix} A \tau^{-1/2} & B \tau^{-1/2} \\ C \tau^{-1/2} & D \tau^{-1/2} \end{bmatrix}^s. \quad (3.68)$$

Now, Sylvester's theorem Eq. (3.37) may be applied to the matrix on the right-hand side since it is now unimodular:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \frac{\tau^{(s-1)/2}}{\sin \theta'} \begin{bmatrix} A \sin(s \theta') - \tau^{1/2} \sin[(s-1) \theta'] & B \sin(s \theta') \\ C \sin(s \theta') & D \sin(s \theta') - \tau^{1/2} \sin[(s-1) \theta'] \end{bmatrix}. \quad (3.69)$$

The angle θ' is defined by the relationship

$$\cos \theta' \equiv \frac{1}{2}(A + D) \tau^{-1/2}. \quad (3.70)$$

Birefringent optical systems often include the use of polarizers. In filter design, for example, polarizers are used to discard unwanted frequency components. In the Jones calculus, polarizers are represented by zero determinant matrices. Since the determinant of a product is the product of the determinants, it follows that any optical system that

includes polarizers would be represented by a zero determinant matrix. Thus it is important to derive a zero determinant form of Sylvester's theorem. From Eq. (3.34) it follows that the eigenvalues are $A+D$, and zero, and from Eq. (3.32) it is seen that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = (A + D)^{s-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (3.71)$$

The results of this subsection are summarized in Table II. Due to the generality of the nonunimodular form of Sylvester's theorem, it is appropriate to examine special cases. Thus several specific matrix operations which have different exponents are also identified in Table II. Other matrix operations which may be derived directly from Sylvester's theorem may also be derived using the special case matrices. For example, the square root matrix in Table II may be applied to itself to yield the fourth root matrix operation:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{1/4} = \frac{1^{1/4}}{\delta^{1/2}(2\tau^{1/4} + \delta^{1/2})^{1/2}} \begin{bmatrix} A + \tau^{1/2} + \tau^{1/4}\delta^{1/2} & B \\ C & D + \tau^{1/2} + \tau^{1/4}\delta^{1/2} \end{bmatrix}, \quad (3.72)$$

where

$$\delta \equiv A + D + 2\tau^{1/2}. \quad (3.73)$$

The determinant of the original matrix is $\tau = AD - BC$. Thus the determinant of the matrix in Eq. (3.72) is $\tau^{1/4}$. It is important to note that according to Sylvester's theorem, a matrix taken to the -1 power is equivalent to the matrix inverse, and a matrix taken to the zero power is the identity matrix, as expected.

TABLE II
GENERALIZED 2X2 SYLVESTER THEOREMS

Description	Operation	Matrix $\tau \equiv AD - BC$ $\cos\theta \equiv \frac{1}{2}(A + D)\tau^{-1/2}$
Sylvester's Theorem $\tau \neq 0$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s$	$\frac{\tau^{(s-1)/2}}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \tau^{1/2} \sin[(s-1)\theta] & B \sin(s\theta) \\ C \sin(s\theta) & D \sin(s\theta) - \tau^{1/2} \sin[(s-1)\theta] \end{bmatrix}$
Sylvester's Theorem $\tau = 1$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s$	$\frac{1}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \sin[(s-1)\theta] & B \sin(s\theta) \\ C \sin(s\theta) & D \sin(s\theta) - \sin[(s-1)\theta] \end{bmatrix}$
Sylvester's Theorem $\tau = 0$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s$	$(A + D)^{s-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$
Squared Matrix	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^2$	$\begin{bmatrix} A(A+D) - \tau & B(A+D) \\ C(A+D) & D(A+D) - \tau \end{bmatrix}$
Unit Matrix	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^1$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$
Square Root Matrix	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{1/2}$	$\frac{\pm 1}{[A+D+2\tau^{1/2}]^{1/2}} \begin{bmatrix} A + \tau^{1/2} & B \\ C & D + \tau^{1/2} \end{bmatrix}$
Identity Matrix	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^0$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Inv. Root Matrix	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1/2}$	$\frac{\pm 1}{\tau^{1/2}[A+D+2\tau^{1/2}]^{1/2}} \begin{bmatrix} D + \tau^{1/2} & -B \\ -C & A + \tau^{1/2} \end{bmatrix}$
Inv. Matrix	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}$	$\frac{1}{\tau} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$
Inv. Squared Matrix	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-2}$	$\frac{1}{\tau^2} \begin{bmatrix} D(A+D) - \tau & -B(A+D) \\ -C(A+D) & A(A+D) - \tau \end{bmatrix}$

Nonunimodular 3x3 Sylvester theorem

The 2x2 Sylvester's theorems given apply to a wide variety of problems in optics and physics in general. However, as alluded to earlier, there are important cases where a 3x3 matrix is needed. For example, with the 3x3 theory, one may trace light rays and Gaussian beams through misaligned optical systems. Therefore there exists a need for a 3x3 version of Sylvester's theorem for the previous three cases: nonunimodular, unimodular, and zero determinant matrices. The general 9-element 3x3 form of Sylvester's theorem is not necessary, and we are concerned with 3x3 matrices that have the form

$$\begin{bmatrix} X_s \\ Y_s \\ 1 \end{bmatrix} = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^s \begin{bmatrix} X_0 \\ Y_0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} A_s & B_s & E_s \\ C_s & D_s & F_s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ 1 \end{bmatrix}. \quad (3.74)$$

Regardless of the determinant, the A_s , B_s , C_s , and D_s terms of the s th power of the 3x3 matrix in Eq. (3.74) are the same as their 2x2 equivalent Eq. (3.37). Thus, it is only required to find the E_s and F_s elements. As before, one may use Sylvester's matrix polynomial theorem Eq. (3.29) directly, or generalize the derivation given in Appendix A of Ref. 133. Alternately, one may use the commutativity requirement $T^s T = T T^s$ to obtain two equations with two unknowns:

$$A_s E + B_s F + E_s = A E_s + B F_s + E, \quad (3.75)$$

$$C_s E + D_s F + F_s = C E_s + D F_s + F. \quad (3.76)$$

Solving these two equations for E_s and F_s for a unimodular matrix yields

$$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^s = \frac{1}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \sin[(s-1)\theta] & B \sin(s\theta) & E'_s \\ C \sin(s\theta) & D \sin(s\theta) - \sin[(s-1)\theta] & F'_s \\ 0 & 0 & \sin\theta \end{bmatrix}. \quad (3.77)$$

where θ is defined here as in Eq. (3.43) and

$$E'_s \equiv \left[\frac{(A-1)\sin(s\theta) + (D-1)\{\sin[(s-1)\theta] + \sin\theta\}}{2(\cos\theta - 1)} \right] E$$

$$+ \left[\frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{2(\cos\theta - 1)} \right] BF, \quad (3.78)$$

$$F'_s \equiv \left[\frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{2(\cos\theta - 1)} \right] CE \\ + \left[\frac{(D-1)\sin(s\theta) + (A-1)/\sin[(s-1)\theta] + \sin\theta}{2(\cos\theta - 1)} \right] F. \quad (3.79)$$

In a manner similar to the previous subsection, the determinant may be factored out and Eqs. (3.77) - (3.79) may be used to obtain Sylvester's theorem for a nonunimodular 3x3 matrix:

$$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^s = \frac{\tau^{(s-1)/2}}{\sin\theta'} \begin{bmatrix} A \sin(s\theta') - \tau^{1/2}\sin[(s-1)\theta'] & B \sin(s\theta') & E'_s \\ C \sin(s\theta') & D \sin(s\theta') - \tau^{1/2}\sin[(s-1)\theta'] & F'_s \\ 0 & 0 & \sin\theta' \end{bmatrix}. \quad (3.80)$$

As before

$$\cos\theta' \equiv 1/2(A + D)\tau^{-1/2}. \quad (3.81)$$

The elements E'_s and F'_s are

$$E'_s \equiv \left[\frac{(A\tau^{-1/2}-1)\sin(s\theta') + (D\tau^{-1/2}-1)/\sin[(s-1)\theta'] + \sin\theta'}{2(\cos\theta' - 1)} \right] E \\ + \left[\frac{\sin(s\theta') - \sin[(s-1)\theta'] - \sin\theta'}{2\tau^{1/2}(\cos\theta' - 1)} \right] BF, \quad (3.82)$$

$$F'_s \equiv \left[\frac{\sin(s\theta') - \sin[(s-1)\theta'] - \sin\theta'}{2\tau^{1/2}(\cos\theta' - 1)} \right] CE \\ + \left[\frac{(D\tau^{-1/2}-1)\sin(s\theta') + (A\tau^{-1/2}-1)/\sin[(s-1)\theta'] + \sin\theta'}{2(\cos\theta' - 1)} \right] F. \quad (3.83)$$

The zero-determinant form of the 3x3 Sylvester's theorem may be obtained in a similar manner. The result is

$$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^s = (A + D)^{s-1} \begin{bmatrix} A & B & E'_s \\ C & D & F'_s \\ 0 & 0 & (A+D)^{1-s} \end{bmatrix}, \quad (3.84)$$

where

$$E'_s \equiv \frac{1}{(A + D)^{s-1}} \left\{ E + \left[\frac{(A + D)^{s-1} - 1}{A + D - 1} \right] (AE + BF) \right\}, \quad (3.85)$$

$$F'_s \equiv \frac{1}{(A + D)^{s-1}} \left\{ F + \left[\frac{(A + D)^{s-1} - 1}{A + D - 1} \right] (CE + DF) \right\}. \quad (3.86)$$

These three 3x3 forms of Sylvester's theorem along with an inverse matrix are summarized in Table III.

TABLE III
GENERALIZED 3X3 SYLVESTER THEOREMS

Description	Operation	Matrix
		$\tau \equiv AD - BC$ $\gamma \equiv A + D - 2\tau^{1/2}$ $\cos\theta \equiv 1/2(A + D)\tau^{-1/2}$
Sylvester's Theorem ($\tau \neq 0$)	$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^s$	$\frac{\tau^{(s-1)/2}}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \tau^{1/2} \sin[(s-1)\theta] & B \sin(s\theta) & E'_s \\ C \sin(s\theta) & D \sin(s\theta) - \tau^{1/2} \sin[(s-1)\theta] & F'_s \\ 0 & 0 & \tau^{(1-s)/2} \sin\theta \end{bmatrix}$
Sylvester's Theorem ($\tau = 1$)	$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^s$	$\frac{1}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \sin[(s-1)\theta] & B \sin(s\theta) & E'_s \\ C \sin(s\theta) & D \sin(s\theta) - \sin[(s-1)\theta] & F'_s \\ 0 & 0 & \sin\theta \end{bmatrix}$
Sylvester's Theorem ($\tau = 0$)	$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^s$	$(A + D)^{s-1} \begin{bmatrix} A & B & E'_s \\ C & D & F'_s \\ 0 & 0 & (A + D)^{1-s} \end{bmatrix}$
Inverse Matrix	$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}^{-1}$	$\frac{1}{\tau} \begin{bmatrix} D & -B & \tau^{-1}(BF - DE) \\ -C & A & \tau^{-1}(CE - AF) \\ 0 & 0 & \tau \end{bmatrix}$
$\tau \neq 0$ $E'_s \equiv \left[\frac{(A\tau^{-1/2} - 1)\sin(s\theta) + (D\tau^{-1/2} - 1)/\sin[(s-1)\theta] + \sin\theta}{\gamma\tau^{-1/2}} \right] E + \left[\frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{\gamma} \right] BF$ $F'_s \equiv \left[\frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{\gamma} \right] CE + \left[\frac{(D\tau^{-1/2} - 1)\sin(s\theta) + (A\tau^{-1/2} - 1)/\sin[(s-1)\theta] + \sin\theta}{\gamma\tau^{-1/2}} \right] F$ $\tau = 1$ $E'_s \equiv \left[\frac{(A - 1)\sin(s\theta) + (D - 1)/\sin[(s-1)\theta] + \sin\theta}{\gamma} \right] E + \left[\frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{\gamma} \right] BF$ $F'_s \equiv \left[\frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{\gamma} \right] CE + \left[\frac{(D - 1)\sin(s\theta) + (A - 1)/\sin[(s-1)\theta] + \sin\theta}{\gamma} \right] F$ $\tau = 0$ $E'_s \equiv (A + D)^{1-s} \left\{ E + \left[\frac{(A + D)^{s-1} - 1}{A + D - 1} \right] (AE + BF) \right\}$ $F'_s \equiv (A + D)^{1-s} \left\{ F + \left[\frac{(A + D)^{s-1} - 1}{A + D - 1} \right] (CE + DF) \right\}$		

Again, there is an interest in special cases of the theorem, and the 3x3 extension of Eq. (3.49) is

$$\begin{bmatrix} \cos\theta & \chi\sin\theta & E \\ -\chi^{-1}\sin\theta & \cos\theta & F \\ 0 & 0 & 1 \end{bmatrix}^s = \begin{bmatrix} \cos(s\theta) & \chi\sin(s\theta) & E_s \\ -\chi^{-1}\sin(s\theta) & \cos(s\theta) & F_s \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.87)$$

where

$$E_s \equiv \frac{1 + \cos\theta}{2} \left[\frac{\sin(s\theta)}{\sin\theta} + \frac{1 - \cos(s\theta)}{1 + \cos\theta} \right] E + \frac{\chi\sin\theta}{2} \left[\frac{1 - \cos(s\theta)}{1 - \cos\theta} - \frac{\sin(s\theta)}{\sin\theta} \right] F, \quad (3.88)$$

$$F_s \equiv \frac{1 + \cos\theta}{2} \left[\frac{\sin(s\theta)}{\sin\theta} + \frac{1 - \cos(s\theta)}{1 + \cos\theta} \right] F - \frac{\chi^{-1}\sin\theta}{2} \left[\frac{1 - \cos(s\theta)}{1 - \cos\theta} - \frac{\sin(s\theta)}{\sin\theta} \right] E. \quad (3.89)$$

Two important special cases of this result occur when $\theta \rightarrow 0$:

$$\begin{bmatrix} 1 & B & E \\ 0 & 1 & F \\ 0 & 0 & 1 \end{bmatrix}^s = \begin{bmatrix} 1 & sB & sE + \frac{1}{2}s(s-1)BF \\ 0 & 1 & sF \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.90)$$

$$\begin{bmatrix} 1 & 0 & E \\ C & 1 & F \\ 0 & 0 & 1 \end{bmatrix}^s = \begin{bmatrix} 1 & 0 & sE + \frac{1}{2}s(s-1)CE \\ 0 & 1 & sF \\ C & 0 & 1 \end{bmatrix}. \quad (3.91)$$

Here the definition of χ ($\chi\sin\theta \rightarrow B$, $-\chi^{-1}\sin\theta \rightarrow C$) has been used.

SUMMARY

Backward propagation through an optical system occurs in a large class of multipass applications where the optical signal traverses the system at least twice. Such applications include remote sensing, nondestructive evaluation, and synthesis of optical delay lines and laser oscillators. In the ubiquitous matrix theories considered here, a 2x2 transfer matrix is used to represent forward propagation of light through the optical system, and a corresponding reverse matrix is used to represent backward propagation. A general procedure has been demonstrated to obtain reverse matrices. The reverse matrix has been found for several generalized theories which make use of matrices which may be nonunimodular, possess an important 3x3 form, or both. A possible application of the results has been demonstrated with an example. In particular, it was shown that, by tilting a mirror, a laser may be aligned even though it possesses misaligned intracavity optics.

When Sylvester's theorem is applied in unimodular 2×2 transfer matrix optics, it governs light propagation through periodic optical systems. There are numerous physically interesting and important applications of these periodic systems. However, some applications such as those involving periodic distributed feedback lasers, lossy birefringent filters, electric circuits with intranetwork independent sources, high energy particle accelerators, periodic computer graphics manipulations with object translations, and periodic pulse compressors, require nonunimodular and/or an augmented 3×3 matrix formalism. Sylvester's theorem has been extended here to facilitate the study of these and other systems. Systematic procedures have also been used to find the range of validity of Sylvester's theorem, and several important special cases have been identified. For example, roots of matrices, which are useful for system synthesis, were examined. The basic results have been summarized in tabular form, and it is straightforward to extend these results to other types of matrices.

CHAPTER IV

LASER ELECTROMAGNETICS

INTRODUCTION

A substantial portion of laser optics may be treated using transfer matrix methods. These matrix methods are usually derived from Maxwell's equations. The purpose of this chapter is review some of the fundamental concepts regarding Maxwell's equations and laser electromagnetics in general. This will provide some of the mathematical and physical tools to derive these matrix methods. Of particular interest is the application of laser electromagnetic theory to laser amplifiers. Some important results are derived, while others are reviewed.

MAXWELL'S EQUATIONS

In a landmark paper James Clerk Maxwell (1831-1879) summarized the laws of electricity and magnetism as a set of differential equations [209]. This alone was a significant accomplishment, but he also brilliantly showed that an important term in one of the differential equations was missing. This factor is now known as the Maxwell displacement current. With the displacement current, these equations summarize the theory of the electric and magnetic fields, similar to Newton's laws which summarize gravitational field effects.

Maxwell's theory had some startling ramifications. With the displacement current, he showed that a time-varying electric field creates a magnetic field, which creates an electric field, which creates a magnetic field, ad infinitum. This series of field interac-

tions is known as an electromagnetic wave. Thus an accelerating point charge, for example, loses some of its energy which goes into the creation of an electromagnetic wave, i.e., it radiates. The speed of the wave was calculated in terms of known electric and magnetic constants, and the result was within experimental error of the then measured speed of light. For this and other reasons, the conclusion was inescapable: light is an electromagnetic wave. In Maxwell's own words [209]:

The general equations are next applied to the case of a magnetic disturbance through a non-conducting field, and it is shown that the only disturbances which can be so propagated are those which are transverse to the direction of propagation, and that the velocity of propagation is the velocity v , found from experiments such as those of Weber, which expresses the number of electrostatic units of electricity which are contained in one electromagnetic unit.

This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws...

Maxwell had his detractors, and his theory was not immediately accepted. Some researchers had worked their entire lives on different theories of light propagation, and they weren't about to give their theories up without substantial proof. As a sad note in the history of science, Maxwell died in 1879 before proof came. However, he was not alone in the promotion of his theory. Several "Maxwellians" such as the electrical engineer Oliver Heaviside (1850-1925) and the physicist John H. Poynting (1852-1914) also championed his theory.

Maxwell originally wrote 20 awkward equations with 20 variables. These equations also contained potential functions and quaternions. The modern form of Maxwell's equations were first written by Heaviside in 1883. Heaviside abandoned the potential functions in favor of field quantities, and the now archaic quaternions were replaced with vectors and tensors. Maxwell's original equations had no symmetry, but Heaviside's equations contained stunning symmetry. He called his equations "the duplex form of Maxwell's equations." For some years the reformulated equations were called the

"Hertz-Heaviside" equations. Later the young Einstein referred to them as the "Maxwell-Hertz" equations, leaving out the sometimes gruff Heaviside [210]. Heinrich Hertz (1857-1894) wrote Maxwell's equations in a form similar to Heaviside's duplex equations, but he acknowledges Heaviside's priority in writing them in that form. It seems curious to call a set of equations which Maxwell may never have recognized "Maxwell's equations", their modern name. Considering this historical background, it seems appropriate to call them the "Maxwell-Heaviside equations", or perhaps the "Heaviside form of Maxwell's equations."

It was Hertz who finally proved the validity of Maxwell's equations by creating some of Maxwell's electromagnetic waves, and demonstrating in a series of experiments in 1888 that the waves had many of the properties of light. Thus, Maxwell's place in history (as well as Hertz's) was secured. Indeed, the Maxwell-Heaviside equations represent the basis of all modern electrical engineering.

Heaviside's Integral Form

The Maxwell-Heaviside Equations come in two mathematical forms: the integral form and the differential form. Heaviside's integral form of Maxwell's equations using the right-hand rule convention in the International System of Units (abbreviated SI from the French "Le Systeme International d'Unites") is:

$$\oint_C \vec{E} \cdot d\vec{l} = - \frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} \quad (4.1)$$

$$\oint_C \vec{H} \cdot d\vec{l} = \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S} + \int_S \vec{J} \cdot d\vec{S} \quad (4.2)$$

$$\oiint_S \vec{D} \cdot d\vec{S} = \iiint_V \rho \, dV \quad (4.3)$$

$$\oiint_S \vec{B} \cdot d\vec{S} = 0 \quad (4.4)$$

The vector \vec{E} will be referred to here as the electric Field. The unit of the electric Field is V/m . The Volt, V , is named after Allessandro Volta. In terms of meters (m), kilograms (kg), seconds (s), and Amperes (A), a Volt is a $kg\ m^2\ A^{-1}\ s^{-3}$. These four units when used as a basis set, are sometimes referred to as rationalized mksa units. The Ampere was named after Andre Marie Ampere (1775-1836). The ratio of a Volt to an Ampere is an Ohm, Ω . The Ohm is named after Georg Simon Ohm (1787-1854), a German physicist. The vector \vec{B} is the magnetic flux density whose units are Wb/m^2 . A Weber, Wb , is a $V\ s$ which is a $kg\ m^2\ A^{-1}\ s^{-2}$. The unit was named after Wilhelm Weber. The electric current density is \vec{J} . It's unit is A/m^2 . The electric displacement density is \vec{D} , and has units of $A\ s\ m^{-2}$. The magnetic field, \vec{H} , has a unit of A/m .

The first of these equations, named after the English chemist and physicist Michael Faraday (1791-1867), is the integral form of Faraday's Induction law and has units of volts. Faraday's law in integral form may be summarized as follows: The electromotive force around a closed path C is the negative of the time rate of change of the magnetic flux through an open surface S which is bounded by C . Thus a time-dependent magnetic field through a closed loop induces an electric field in the vicinity of the loop. Alternately, an electric field in a closed loop induces a magnetic field through the loop. The negative sign in Faraday's law is associated with "Lenz's law" which insures energy conservation.

Equation (4.2) is Ampere's circuital law and has units of Amperes. It may be summarized as follows: the magnetomotive force around a closed path C is the sum of the current due to charges, and the Maxwell displacement current through an open surface S which is bounded by C . Thus either a time-dependent electric field or a current through a closed loop induces a magnetic field in the vicinity of the loop.

Equation (4.3) is known as Gauss's law for the electric field and has units of Coulombs. Gauss's law for the electric field in integral form may be summarized as

follows: the displacement flux crossing a closed surface S is equal to the net charge in the volume V enclosed by S . Thus charge is a source of the electric field.

Karl F. Gauss (1777-1855) was the German mathematician and physicist for whom Eq. (4.3) and Eq. (4.4), "Gauss's law for the magnetic field", were named. The law has units of Webers. Gauss's law for the magnetic field in integral form may be summarized as follows: The magnetic flux crossing a closed surface S is zero. Thus there are no sources (known as magnetic monopoles) or sinks of the magnetic field. Put another way, all magnetic field lines are closed loops.

An equation associated with the Maxwell-Heaviside equations is the continuity equation which is also known as the conservation of charge:

$$\oint_S \vec{J} \cdot d\vec{S} = - \frac{\partial}{\partial t} \iiint_V \rho dV . \quad (4.5)$$

The electric charge density is ρ , and its unit is C/m^3 which in mksa units is an $A \cdot s \cdot m^{-3}$. The unit Coulomb, C , was named after the French scientist Charles Augustin de Coulomb (1736-1806). Thus if the amount of charge contained in a volume is changing, a current must be crossing the surface bounding the volume.

It should be noted that these five equations are not independent. Equations (4.1) and (4.2) together with either Eq. (4.3) or (4.5) form an independent set. For example, the continuity equation may be derived from Eqs. (4.2) and (4.3) in the following manner: consider Ampere's law Eq. (4.2) in the limit that the closed curve C become infinitely small. The left hand side of Eq. (4.2) becomes zero, and the surface integrals on the right hand side of the equation become closed surface integrals. Substituting Gauss's law for the electric field Eq. (4.3) into this equation results in Eq. (4.5), the conservation of electric charge.

The Maxwell-Heaviside Equations in integral form contain various line and surface integrals. The differential elements, $d\vec{l}$, $d\vec{S}$, and dV vary depending on the coordinate

system used. Table IV lists these differential elements for three coordinate systems of interest.

TABLE IV
DIFFERENTIAL LENGTH, SURFACE, AND VOLUME ELEMENTS

Coordinate System	$d\vec{l}$	$\pm d\vec{S}$	dV
Rectangular (x, y, z)	$dx\vec{i}_x + dy\vec{i}_y + dz\vec{i}_z$	$dydz\vec{i}_x$ $dzdx\vec{i}_y$ $xdy\vec{i}_z$	$dx dy dz$
Cylindrical (r, ϕ, z)	$dr\vec{i}_r + r d\phi \vec{i}_\phi + dz\vec{i}_z$	$r d\phi dz\vec{i}_r$ $dr dz\vec{i}_\phi$ $r dr d\phi \vec{i}_z$	$r dr d\phi dz$
Spherical (R, θ, ϕ) ($0 \leq \theta \leq \pi$)	$dR\vec{i}_R + R d\theta \vec{i}_\theta + R \sin\theta d\phi \vec{i}_\phi$	$R^2 \sin\theta d\theta d\phi \vec{i}_R$ $R \sin\theta dR d\phi \vec{i}_\theta$ $R dR d\theta \vec{i}_\phi$	$R^2 \sin\theta dR d\theta d\phi$

It is often of interest to change from one coordinate system to another, and Table V is useful to this end.

TABLE V
COORDINATE SYSTEM CONVERSION FORMULAS

Coordinate System	Scalar Conversions	Vector Conversions
Cylindrical (r, ϕ, z)	$x = r \cos \phi$ $y = r \sin \phi$ $z = z$	$\bar{i}_x = \cos \phi \bar{i}_r - \sin \phi \bar{i}_\phi$ $\bar{i}_y = \sin \phi \bar{i}_r + \cos \phi \bar{i}_\phi$ $\bar{i}_z = \bar{i}_z$
	$r = (x^2 + y^2)^{1/2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\bar{i}_r = \cos \phi \bar{i}_x + \sin \phi \bar{i}_y$ $\bar{i}_\phi = -\sin \phi \bar{i}_x + \cos \phi \bar{i}_y$ $\bar{i}_z = \bar{i}_z$
Spherical (R, θ, ϕ) ($0 \leq \theta \leq \pi$)	$x = R \sin \theta \cos \phi$ $y = R \sin \theta \sin \phi$ $z = R \cos \theta$	$\bar{i}_x = \sin \theta \cos \phi \bar{i}_R + \cos \theta \cos \phi \bar{i}_\theta - \sin \phi \bar{i}_\phi$ $\bar{i}_y = \sin \theta \sin \phi \bar{i}_R + \cos \theta \sin \phi \bar{i}_\theta + \cos \phi \bar{i}_\phi$ $\bar{i}_z = \cos \theta \bar{i}_R - \sin \theta \bar{i}_\theta$
	$R = (x^2 + y^2 + z^2)^{1/2}$ $\theta = \tan^{-1}[(x^2 + y^2)^{1/2}/z]$ $\phi = \tan^{-1}(y/x)$	$\bar{i}_R = \sin \theta \cos \phi \bar{i}_x + \sin \theta \sin \phi \bar{i}_y + \cos \theta \bar{i}_z$ $\bar{i}_\theta = \cos \theta \cos \phi \bar{i}_x + \cos \theta \sin \phi \bar{i}_y - \sin \theta \bar{i}_z$ $\bar{i}_\phi = -\sin \phi \bar{i}_x + \cos \phi \bar{i}_y$

Heaviside's Differential Form

Given Heaviside's integral form of Maxwell's equations it is possible to derive a differential form with the use of two mathematical theorems. For any vector \bar{A} ,

$$\iint_S (\bar{\nabla} \times \bar{A}) \cdot d\bar{S} = \oint_C \bar{A} \cdot d\bar{l} \quad , \quad (4.6)$$

$$\iiint_V (\bar{\nabla} \cdot \bar{A}) dV = \oiint_S \bar{A} \cdot d\bar{S} \quad . \quad (4.7)$$

The first of these is Stokes' theorem named after George Gabriel Stokes (1819-1903). The second is Gauss's theorem, though it is commonly called the divergence theorem. Alternate names are Stokes' curl theorem and Gauss's divergence theorem since " $\bar{\nabla} \times$ " is the curl operator and " $\bar{\nabla} \cdot$ " is the divergence operator.

Heaviside's differential form of the Maxwell's equations using the right hand rule convention in the international system of units is

$$\bar{\nabla} \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad (4.8)$$

$$\bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \quad (4.9)$$

$$\bar{\nabla} \cdot \bar{D} = \rho \quad (4.10)$$

$$\bar{\nabla} \cdot \bar{B} = 0 \quad (4.11)$$

The units of Faraday's law, Eq. (4.8), is Volts per square meter (V/m^2), or a $kg \ A^{-1} \ s^{-3}$.

The units of Ampere's law, Eq. (4.9), is Amperes per square meter (A/m^2). The units of

Gauss's law for the electric field, Eq. (4.10), are Coulombs per cubic meter (C/m^3), or

Ampere-seconds per cubic meter (As/m^3). The units of Gauss's law for the magnetic

field, Eq. (4.11), are Webers per cubic meter (Wb/m^3).

Here, the associated continuity equation is

$$\bar{\nabla} \cdot \bar{J} = - \frac{\partial \rho}{\partial t} \quad (4.12)$$

The units of this equation are Amperes per cubic meter (A/m^3).

Unlike the integral form, the differential form of the Maxwell-Heaviside equations requires a mathematical statement of the boundary conditions. The boundary conditions can be derived from the integral form. The boundary conditions are

$$\bar{i}_n \times (\bar{E}_2 - \bar{E}_1) = \bar{0} \quad (4.13)$$

$$\bar{i}_n \times (\bar{H}_2 - \bar{H}_1) = -\bar{J}_s \quad (4.14)$$

$$\bar{i}_n \cdot (\bar{D}_2 - \bar{D}_1) = -\rho_s \quad (4.15)$$

$$\bar{i}_n \cdot (\bar{B}_2 - \bar{B}_1) = 0 \quad (4.16)$$

where \bar{i}_n is the unit vector normal to the surface boundary in the direction of medium 1.

Thus, from Eq. (4.13), the electric field components tangential to a boundary are continuous across that boundary. From Eq. (4.14), the tangential magnetic field components are discontinuous across a boundary by the amount of the current density flowing on the boundary. From Eq. (4.15), the electric displacement component normal to a boundary is discontinuous across it by the amount of the charge density on the boundary. From Eq. (4.16), the normal component of the magnetic flux density is continuous across a boundary.

With the differential form of the Maxwell-Heaviside equations, the differential operators in different coordinate systems are of interest:

Vector Differential Operations [Rectangular Coordinates (x, y, z)]

$$\begin{aligned}\nabla\psi &= \frac{\partial\psi}{\partial x}\bar{i}_x + \frac{\partial\psi}{\partial y}\bar{i}_y + \frac{\partial\psi}{\partial z}\bar{i}_z \\ \nabla \cdot \bar{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ \nabla \times \bar{A} &= \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] \bar{i}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right] \bar{i}_y + \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right] \bar{i}_z \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \\ \nabla^2\bar{A} &= (\nabla^2 A_x)\bar{i}_x + (\nabla^2 A_y)\bar{i}_y + (\nabla^2 A_z)\bar{i}_z\end{aligned}$$

Vector Differential Operations [Cylindrical Coordinates (r, ϕ, z)]

$$\begin{aligned}\nabla\psi &= \frac{\partial\psi}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\bar{i}_\phi + \frac{\partial\psi}{\partial z}\bar{i}_z \\ \nabla \cdot \bar{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\ \nabla \times \bar{A} &= \left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} \right] \bar{i}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right] \bar{i}_\phi + \left[\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\phi} \right] \bar{i}_z\end{aligned}$$

$$\nabla^2\psi = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$\nabla^2 \bar{A} = \left[\nabla^2 A_r - \frac{2}{r^2} \frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2} \right] \bar{i}_r + \left[\nabla^2 A_\phi + \frac{2}{r^2} \frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2} \right] \bar{i}_\phi + (\nabla^2 A_z) \bar{i}_z$$

Vector Differential Operations [Spherical Coordinates (R, θ, ϕ)]

$$\nabla \psi = \frac{\partial \psi}{\partial R} \bar{i}_R + \frac{1}{R} \frac{\partial \psi}{\partial \theta} \bar{i}_\theta + \frac{1}{R \sin \theta} \frac{\partial \psi}{\partial \phi} \bar{i}_\phi$$

$$\nabla \cdot \bar{A} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_r) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\begin{aligned} \nabla \times \bar{A} = & \frac{1}{R \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] \bar{i}_R + \frac{1}{R} \left[\frac{1}{\sin \theta} \frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R} (R A_\phi) \right] \bar{i}_\theta \\ & + \frac{1}{R} \left[\frac{\partial (R A_\theta)}{\partial R} - \frac{\partial A_R}{\partial \theta} \right] \bar{i}_\phi \end{aligned}$$

$$\nabla^2 \psi = \frac{1}{R^2} \frac{\partial}{\partial R} \left[R^2 \frac{\partial \psi}{\partial R} \right] + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial \psi}{\partial \theta} \right] + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

$$\begin{aligned} \nabla^2 \bar{A} = & \left[\nabla^2 A_R - \frac{2}{R^2} \left[A_R + \cot \theta A_\theta + \csc \theta \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_\theta}{\partial \theta} \right] \right] \bar{i}_R \\ & + \left[\nabla^2 A_\theta - \frac{1}{R^2} \left[\csc^2 \theta A_\theta - 2 \frac{\partial A_R}{\partial \theta} + 2 \cot \theta \csc \theta \frac{\partial A_\phi}{\partial \phi} \right] \right] \bar{i}_\theta \\ & + \left[\nabla^2 A_\phi - \frac{1}{R^2} \left[\csc^2 \theta A_\phi - 2 \csc \theta \frac{\partial A_R}{\partial \phi} - 2 \cot \theta \csc \theta \frac{\partial A_\theta}{\partial \phi} \right] \right] \bar{i}_\phi \end{aligned}$$

In addition there are several formulas of vector calculus used in laser electromagnetics:

$$\nabla(\psi + \Phi) = \nabla\psi + \nabla\Phi$$

$$\nabla \cdot (\bar{A} + \bar{B}) = \nabla \cdot \bar{A} + \nabla \cdot \bar{B}$$

$$\nabla \times (\bar{A} + \bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B}$$

$$\nabla(\psi\Phi) = \psi\nabla\Phi + \Phi\nabla\psi$$

$$\nabla \cdot (\psi \bar{A}) = \psi \nabla \cdot \bar{A} + \bar{A} \cdot \nabla \psi$$

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \times \bar{A} - \bar{A} \cdot \nabla \times \bar{B}$$

$$\nabla \times (\psi \bar{A}) = \psi \nabla \times \bar{A} + \nabla \psi \times \bar{A}$$

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{A} \nabla \cdot \bar{B} - \bar{B} \nabla \cdot \bar{A} + (\bar{B} \cdot \nabla) \bar{A} - (\bar{A} \cdot \nabla) \bar{B}$$

$$\nabla \cdot \nabla \psi \equiv \nabla^2 \psi$$

$$\nabla \cdot \nabla \times \bar{A} = 0$$

$$\nabla \times \nabla \phi = 0$$

$$\nabla^2 \bar{A} \equiv \nabla \times \nabla \times \bar{A} - \nabla(\nabla \cdot \bar{A})$$

$$\nabla(\bar{A} \cdot \bar{B}) = (\bar{A} \cdot \nabla) \bar{B} + (\bar{B} \cdot \nabla) \bar{A} + \bar{A} \times (\nabla \times \bar{B}) + \bar{B} \times (\nabla \times \bar{A})$$

$$\bar{A} \cdot \bar{B} \times \bar{C} = \bar{B} \cdot \bar{C} \times \bar{A} = \bar{C} \cdot \bar{A} \times \bar{B}$$

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$$

The Constitutive Relations

The Maxwell-Heaviside equations themselves do not contain sufficient information to solve any given problem. Additional material relations (also known as "constitutive relations") must be specified. In freespace, the constitutive relations are

$$\bar{D} = \epsilon_o \bar{E} \quad , \quad (4.17)$$

$$\bar{B} = \mu_o \bar{H} \quad , \quad (4.18)$$

$$\bar{J} = \bar{0} \quad . \quad (4.19)$$

The permeability of freespace, μ_o , is exactly $4\pi \times 10^{-7} H/m$. The Henry, H , is named after Joseph Henry (1797-1878). The permittivity of freespace, ϵ_o , is related to the vacuum speed of light by the relation

$$c = (\mu_o \epsilon_o)^{-1/2} \quad . \quad (4.20)$$

Since 1983 when the 17th Conference Generale des Poids et Mesures in Paris redefined the meter in terms of the speed of light, $c = 2.99792458 \times 10^8 \text{ m/s}$, exactly [211]. Thus the permittivity of freespace has an exact value: $\epsilon_o = 10^{-9}/[4\pi(2.99792458)^2] \text{ F/m}$. This is approximately $8.854187 \times 10^{-12} \text{ F/m}$. For analytical calculations the approximation $\epsilon_o \approx 10^{-9}/36\pi \text{ F/m}$ is sometimes used.

The constitutive relations may be much more complicated in different material media. In fact, even in vacuum the simple constitutive relations above [Eqs. (4.17)-(4.19)] apply only approximately. In the atomic and subatomic domain, quantum-mechanical nonlinear effects may be important. However, "in the classical domain of sizes and attainable field strengths there is abundant evidence for the validity of linear superposition and no evidence against it" (as of 1975) [212]. Thus, though this vacuum polarization effect exists in principle, it is invariably ignored.

As an example of the use of Heaviside's integral form of Maxwell's equations, the electric field emanating from a point charge of charge q in freespace is calculated. From Eq. (4.3),

$$\oiint_S \vec{D} \cdot d\vec{S} = q \quad (4.21)$$

A spherical shell of radius R is chosen as the surface S . From Table IV, $d\vec{S} = R^2 \sin\theta d\theta d\phi \vec{i}_R$ and Eq. (4.21) becomes

$$\int_0^{\pi} \int_0^{2\pi} D_R R^2 \sin\theta d\theta d\phi = q \quad (4.22)$$

For symmetry reasons, D_R does not vary in θ or ϕ , and it follows that

$$\vec{D} = \frac{q}{4\pi R^2} \vec{i}_R \quad (4.23)$$

In freespace the constitutive relation Eq. (4.17) may be used, and it follows that

$$\vec{E} = \frac{q}{4\pi\epsilon_o R^2} \vec{i}_R \quad (4.24)$$

which is the well-known formula for the electric field caused by a point charge.

Electric fields and magnetic fields exert forces on particles with charge analogous to gravitational fields exerting forces on particles with mass:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \longleftrightarrow \vec{F} = m\vec{a} \quad , \quad (4.25)$$

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \vec{i}_R \longleftrightarrow \vec{F}_{12} = G \frac{m_1 m_2}{R^2} \vec{i}_R \quad . \quad (4.26)$$

Thus the Lorentz force law is analogous to Newton's 2nd law in mechanics [Eq. (4.25)], and Coulomb's law is analogous to Newton's law of universal gravitation [Eq. (4.26)]. The gravitational constant has the value $G = 6.67259 \times 10^{-11} \text{ N m}^2/\text{kg}^2$. In dealing with problems where Forces due to mass and charge are important, it is prudent to pay special attention to sign convention. Like charges repel each other while masses attract each other.

When the Maxwell-Heaviside equations are applied to moving media, they, like Newton's laws, are valid. However, it was later found that Newton's laws are only correct for matter moving much slower than the speed of light. However, the Maxwell-Heaviside equations are correct even for very fast moving media. In P. Nahin's words [213]

When Einstein found that Newtonian dynamics had to be modified to be compatible with the special theory of relativity, he also found that Maxwell's equations were already relativistically correct. Magnetic effects are, after all, relativistic effects produced by moving charges, and so Maxwell had automatically built relativity into his equations.

Under the most general conditions, the constitutive relations are written:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad , \quad (4.27)$$

$$\vec{B} = \mu_0 \vec{H} + \vec{M} \quad , \quad (4.28)$$

$$\vec{J} = \sigma \vec{E} + \vec{J}_v \quad . \quad (4.29)$$

The material polarization vector \bar{P} represents the net electric dipole moment per unit volume of the medium. The magnetization vector \bar{M} represents the net magnetic dipole moment per unit volume of the medium. The symbol σ is the conductivity of the medium. The non-Ohmic current is represented by the vector \bar{J}_v .

For simple dielectrics,

$$\bar{P} = \epsilon_o \chi_e \bar{E} \quad , \quad (4.30)$$

where χ_e is the electric susceptibility. From Eq. (4.27), it follow that for simple dielectrics

$$\bar{D} = \epsilon_o (1 + \chi_e) \bar{E} \equiv \epsilon_o \epsilon_r \bar{E} \equiv \epsilon \bar{E} \quad . \quad (4.31)$$

The constant ϵ is the permittivity of the material and ϵ_r is the relative permittivity.

Similarly, for simple magnetic materials

$$\bar{M} = \mu_o \chi_m \bar{H} \quad , \quad (4.32)$$

where χ_m is the magnetic susceptibility. From Eq. (4.28), it follow that for simple magnetic materials

$$\bar{B} = \mu_o (1 + \chi_m) \bar{H} \equiv \mu_o \mu_r \bar{H} \equiv \mu \bar{H} \quad . \quad (4.33)$$

The constant μ is the permeability of the material and μ_r is the relative permeability. For simple conductors, $\bar{J}_v = \bar{0}$.

There are several types of complications that may arise in material media: inhomogeneity, anisotropy, nonlinearity, hysteresis, and chirality. Inhomogeneous media have spatially dependent permittivity, permeability, and/or conductivity. Thus, a medium may be electrically inhomogeneous, magnetically inhomogeneous, and/or conductively inhomogeneous. Similarly, anisotropic media have a tensor permittivity, permeability, and/or conductivity. A nonlinear medium has a material polarization, magnetization, and/or non-Ohmic current that depends nonlinearly on the electric field, magnetic field, and/or the electric field, respectively. A material with hysteresis has material properties that

depend on previous values of the field.

THE WAVE EQUATION

To analyze electromagnetic wave propagation, the differential form of the Maxwell-Heaviside equations are combined with the relevant constitutive relations to form a wave equation. The wave equation is a specific reduction of the Maxwell-Heaviside equations to achieve one equation with a single unknown.

For a wide variety of optical media including laser amplifiers, the constitutive relations are

$$\bar{D}(x, y, z, t) = \epsilon(x, y, z) \bar{E}(x, y, z, t) + \bar{P}_{laser}(x, y, z, t) , \quad (4.34)$$

$$\bar{B}(x, y, z, t) = \mu \bar{H}(x, y, z, t) , \quad (4.35)$$

$$\bar{J}(x, y, z, t) = \sigma(x, y, z) \bar{E}(x, y, z, t) . \quad (4.36)$$

where $\epsilon(x, y, z)$ represents the background permittivity of, for example, a laser amplifier. In this case, $\bar{P}_{laser}(x, y, z, t)$ would be the field polarization induced by the lasing atoms. Though the polarization is written as only a function of space and time, it is understood that it may also be a function of the field intensity. Thus, the laser medium is allowed to be electrically nonlinear.

An initial wave equation may be formed by taking the curl of both sides of Heaviside's differential form of Faraday's law [Eq. (4.8)] and using Ampere's law [Eq. (4.9)] and the constitutive relations [Eqs. (4.34)-(4.36)]:

$$\bar{\nabla} \times \bar{\nabla} \times \bar{E} = - \frac{\partial [\bar{\nabla} \times (\mu \bar{H})]}{\partial t} \quad (4.37)$$

$$= - \mu \frac{\partial [\bar{\nabla} \times \bar{H}]}{\partial t} \quad (4.38)$$

$$= - \mu \frac{\partial}{\partial t} \left[\bar{J} + \frac{\partial \bar{D}}{\partial t} \right] \quad (4.39)$$

$$= -\mu \frac{\partial}{\partial t} \left[\sigma \bar{E} + \frac{\partial}{\partial t} \left(\epsilon \bar{E} + \bar{P}_{laser} \right) \right] \quad (4.40)$$

$$= -\mu \sigma \frac{\partial \bar{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \bar{E}}{\partial t^2} - \mu \frac{\partial^2 \bar{P}_{laser}}{\partial t^2} . \quad (4.41)$$

Further reduction of the wave Eq. (4.41) is not possible without foreknowledge of the form of the field polarization. This form may be obtained by postulate, or by rigorous solution. Our interest here is in the latter case, and Schrodinger-based density matrix equations may be used to show that an incident single frequency electric field polarizes a medium in the following manner:

$$\bar{P}_{laser}(x, y, z, t) = \bar{C}(x, y, z, t) \cos(\omega t) + \bar{S}(x, y, z, t) \sin(\omega t) \quad (4.42)$$

$$= \text{Re}[(\bar{C}(x, y, z, t) - i\bar{S}(x, y, z, t))e^{i\omega t}] \quad (4.43)$$

$$= \text{Re}[\epsilon(x, y, z) \chi(x, y, z, t) \bar{E}'(x, y, z, t) e^{i\omega t}] , \quad (4.44)$$

where the susceptibility $\chi(x, y, z, t)$ is complex and the complex amplitude of the electric field, $\bar{E}'(x, y, z, t)$ is defined by

$$\bar{E}(x, y, z, t) = \text{Re}[\bar{E}'(x, y, z, t) e^{i\omega t}] . \quad (4.45)$$

Some of the explicit approximations used in the semiclassical density matrix model for light-matter interactions are

1. Central field
2. Perturbation Hamiltonian
3. Electric dipole transition
4. Two level atoms
5. Phenomenological decay terms
6. Rotating wave approximation
7. Rate equation approximation
8. Dipoles are parallel to field.

In the first approximation, it is assumed that the potential is spherically symmetric which allows one to use parity. It is also assumed that the incident light field is small enough so as not to completely change the eigenstate of the laser medium. This allows use of a perturbation Hamiltonian ($H' \ll H_o$). The form of the perturbation Hamiltonian is assumed to be approximately a pure dipole transition ($H' = -e\vec{r} \cdot \vec{E}$). A phenomenologically generalized two energy level model is used to approximate the many level structure that real laser amplifiers possess. The fifth approximation is a reminder of the limitations of the Maxwell-Schrodinger model. In particular, phenomenological decay terms are used to account for the existence of spontaneous emission. The rate equation approximation allows us to ignore coherence effects. Lastly, it is assumed that the atomic dipoles are parallel to the field polarization.

With these approximations, Eqs. (4.44) and (4.45) may be substituted into the wave Eq. (4.41) and it follows that

$$[\nabla \times \nabla \times \vec{E}'] e^{i\omega t} = -\mu\sigma \left[i\omega \vec{E}' + \frac{\partial \vec{E}'}{\partial t} \right] e^{i\omega t} - \mu\epsilon \frac{\partial^2}{\partial t^2} [(1 + \chi) \vec{E}' e^{i\omega t}] \quad (4.46)$$

where an additional constraint on the electric field has been used. This constraint [the imaginary part of Eq. (4.46)] is required since Eq. (4.45) does not uniquely specify $\vec{E}'(x, y, z, t)$. Before dividing both sides of Eq. (4.46) by $\exp(i\omega t)$, it is convenient to define a complex permittivity:

$$\epsilon'(x, y, z, t) \equiv \epsilon(x, y, z) [1 + \chi(x, y, z, t)] \quad (4.47)$$

With this definition, Eq. (4.46) may be rewritten

$$[\nabla \times \nabla \times \vec{E}'] e^{i\omega t} = -\mu\sigma \left[i\omega \vec{E}' + \frac{\partial \vec{E}'}{\partial t} \right] e^{i\omega t} - \mu \frac{\partial^2}{\partial t^2} [\epsilon' \vec{E}' e^{i\omega t}] \quad (4.48)$$

which reduces to

$$\nabla \times \nabla \times \vec{E}' - (\omega^2 \mu \epsilon' - i\omega \mu \sigma) \vec{E}' = -\mu \left[2i\omega \frac{\partial \epsilon'}{\partial t} + \frac{\partial^2 \epsilon'}{\partial t^2} \right] \vec{E}'$$

$$-\mu \left[\sigma + 2i\omega\epsilon' + 2\frac{\partial\epsilon'}{\partial t} \right] \frac{\partial\bar{E}'}{\partial t} - \mu\epsilon' \frac{\partial^2\bar{E}'}{\partial t^2} . \quad (4.49)$$

It is conventional at this point to use the definition of the vector Laplacian,

$$\nabla^2\bar{E}' \equiv \nabla(\bar{\nabla} \cdot \bar{E}') - \bar{\nabla} \times \bar{\nabla} \times \bar{E}' , \quad (4.50)$$

to transform Eq. (4.49). However, Eq. (4.50) may itself be simplified with the aid of Gauss's law for the electric field, Eq. (4.10):

$$\bar{\nabla} \cdot \bar{D} = \rho = 0 \quad (4.51)$$

With our definition of the complex permittivity and the complex electric field, Eq. (4.51) may be written

$$\bar{\nabla} \cdot (\epsilon'\bar{E}') = 0 \quad (4.52)$$

from which it follows that

$$\bar{\nabla} \cdot \bar{E}' = -\frac{\nabla\epsilon' \cdot \bar{E}'}{\epsilon'} . \quad (4.53)$$

Substituting Eqs. (4.50) and (4.53) into Eq. (4.49), the wave equation becomes

$$\begin{aligned} \nabla^2\bar{E}' + (\omega^2\mu\epsilon' - i\omega\mu\sigma)\bar{E}' = & -\nabla \left[\frac{\nabla\epsilon' \cdot \bar{E}'}{\epsilon'} \right] + \mu \left[2i\omega\frac{\partial\epsilon'}{\partial t} + \frac{\partial^2\epsilon'}{\partial t^2} \right] \bar{E}' \\ & + \mu \left[\sigma + 2i\omega\epsilon' + 2\frac{\partial\epsilon'}{\partial t} \right] \frac{\partial\bar{E}'}{\partial t} + \mu\epsilon' \frac{\partial^2\bar{E}'}{\partial t^2} \end{aligned} \quad (4.54)$$

Rather than use a complex permittivity, we define a complex propagation constant

$$k^2(x,y,z,t) \equiv \omega^2\mu\epsilon'(x,y,z,t) - i\omega\mu\sigma(x,y,z) , \quad (4.55)$$

and the wave equation becomes

$$\begin{aligned} \nabla^2\bar{E}' + k^2\bar{E}' = & -\nabla \left[\frac{(2k\nabla k + i\omega\mu\nabla\sigma) \cdot \bar{E}'}{k^2 + i\omega\mu\sigma} \right] + \frac{2}{\omega} \left[2ik\frac{\partial k}{\partial t} + \frac{1}{\omega} \left(\frac{\partial k}{\partial t} \right)^2 + \frac{k}{\omega} \frac{\partial^2 k}{\partial t^2} \right] \bar{E}' \\ & + \left[-\mu\sigma + 2i\frac{k^2}{\omega} + \frac{4k}{\omega^2} \frac{\partial k}{\partial t} \right] \frac{\partial\bar{E}'}{\partial t} + \left[\frac{k^2 + i\omega\mu\sigma}{\omega^2} \right] \frac{\partial^2\bar{E}'}{\partial t^2} . \end{aligned} \quad (4.56)$$

Without loss of generality, the complex propagation constant may be assumed to have

the form

$$k(x, y, z, t) = k_o(z) - k'(x, y, z, t)/2. \quad (4.57)$$

Similarly, the complex electric field may be written

$$\bar{E}'(x, y, z, t) \equiv \bar{\Psi}(x, y, z, t) \exp \left[-i \int_0^z \kappa_o(z') dz' \right] \quad (4.58)$$

where $\kappa(z)$ is usually equal to $k_o(z)$, or the real part of $k_o(z)$. Substituting Eqs. (4.57) and (4.58) into Eq. (4.56) results in the wave equation

$$\begin{aligned} \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} + \frac{\partial^2 \bar{\Psi}}{\partial z^2} - 2i \kappa_o \frac{\partial \bar{\Psi}}{\partial z} + (k_o^2 - \kappa_o^2) \bar{\Psi} - \left[k_o k' + i \frac{\partial \kappa_o}{\partial z} - k'^2/4 \right] \bar{\Psi} = \\ - \nabla \left[\frac{[2(k_o - k'/2) \nabla(k_o - k'/2) + i \omega \mu \nabla \sigma] \cdot \bar{\Psi}}{k_o^2 - k_o k' + k'^2/4 + i \omega \mu \sigma} \right] \\ + i k_o \left[\frac{[2(k_o - k'/2) \nabla(k_o - k'/2) + i \omega \mu \nabla \sigma] \cdot \bar{\Psi}}{k_o^2 - k_o k' + k'^2/4 + i \omega \mu \sigma} \right] \bar{i}_z \\ + \frac{2}{\omega} \left[-i(k_o - k'/2) \frac{\partial k'}{\partial t} + \frac{1}{4\omega} \left[\frac{\partial k'}{\partial t} \right]^2 - \frac{k_o - k'/2}{2\omega} \frac{\partial^2 k'}{\partial t^2} \right] \bar{\Psi} \\ + \left[-\mu \sigma + \frac{2i(k_o^2 - k_o k' + k'^2/4)}{\omega} - \frac{2}{\omega^2} (k_o - k'/2) \frac{\partial k'}{\partial t} \right] \frac{\partial \bar{\Psi}}{\partial t} \\ + \left[\frac{k_o^2 - k_o k' + k'^2/4 + i \omega \mu \sigma}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2} \end{aligned} \quad (4.59)$$

This wave equation is the fundamental result of the section. Note that if $k(x, y, z, t)$ is constant and if we choose $\kappa_o = k_o$, then a constant vector is a nontrivial solution to Eq. (4.59), and the result represents an ordinary plane wave.

LASER GAIN AND REFRACTIVE INDEX

Given the complex propagation constant for a laser medium, the laser beam's pro-

propagation characteristics may be obtained by subsequent solution of Eq. (4.59). The purpose of this section is to review the forms of the complex propagation constant for different media.

Lasers are often characterized by an index of refraction n , and a gain coefficient g , and a loss coefficient γ . These quantities are related to the complex propagation constant:

$$k(x, y, z, t) \equiv \beta(x, y, z, t) + i\alpha(x, y, z, t) \quad (4.60)$$

$$= \frac{2\pi\nu}{c}n(x, y, z, t) + \frac{i}{2} \left[g(x, y, z, t) - \gamma(x, y, z, t) \right] \quad (4.61)$$

Homogeneous Broadening

In the simplest laser amplifiers, all of the laser atoms have stimulated transitions with the same energy level lifetime and energy difference. In this case, the laser is said to be "homogeneously broadened". There are several effects that contribute to the lifetime, and therefore the linewidth, of the stimulated transition. Firstly, other effects neglected, the laser atoms have a finite lifetime. This lifetime can, in principle, be determined from quantum mechanics. This aspect of homogeneous broadening is referred to as "natural broadening." Another aspect of homogeneous broadening is "collision broadening," which is due to phase-interrupting collisions. When the collisions occur between like atoms, the type of collision broadening is known as "Lorentz broadening." Similarly, when the collisions occur between unlike atoms, the type of collision broadening is "Holtsmark broadening." Collisions may also occur between the laser atoms and phonons. If the collision lifetime due to collision broadening and lifetimes of both of the energy levels which make up the stimulating transition (natural broadening) are known, then the total homogeneous linewidth is

$$\Delta\nu_h = \frac{1}{2\pi\tau_1} + \frac{1}{2\pi\tau_2} + \frac{1}{\pi\tau_{collision}} \quad (4.62)$$

The index of refraction and gain for a homogeneously broadened laser are

$$n_{homogeneous}(x, y, z, t) = n_o(x, y, z, t) + \frac{c g_{ho}(x, y, z, t)}{4\pi v} \frac{\hat{y}}{1 + \hat{y}^2 + sI(x, y, z, t)} \quad (4.63)$$

$$g_{homogeneous}(x, y, z, t) = \frac{g_{ho}(x, y, z, t)}{1 + \hat{y}^2 + sI(x, y, z, t)} \quad (4.64)$$

where

$$\hat{y} \equiv 2(v - v_o)/\Delta v_h \quad (4.65)$$

$$I(x, y, z, t) \equiv \frac{c \epsilon}{2n_o} \bar{\Psi}^*(x, y, z, t) \cdot \bar{\Psi}(x, y, z, t) . \quad (4.66)$$

It may be noticed a spatial or temporal dependence of the unsaturated line-center gain g_{ho} causes a similar refractive index profile.

Mixed Broadening

For single-mode lasers that are not homogeneously broadened, the gain and index of refraction are

$$n_{mixed}(x, y, z, t) = n_o(x, y, z, t) + \frac{c g_{ho}(x, y, z, t)}{4\pi v} \int_0^\infty \frac{\left[\frac{2(v - v_a)}{\Delta v_h} \right] p(v_a) dv_a}{1 + \left[\frac{2(v - v_a)}{\Delta v_h} \right]^2 + sI(x, y, z, t)} \quad (4.67)$$

$$g_{mixed}(x, y, z, t) = g_{ho}(x, y, z, t) \int_0^\infty \frac{p(v_a) dv_a}{1 + \left[\frac{2(v - v_a)}{\Delta v_h} \right]^2 + sI(x, y, z, t)} . \quad (4.68)$$

Basically, if each of the laser atoms have different center frequencies, then the laser acts like many small homogeneously broadened lasers. The distribution of these center frequencies is $p(v_a)$. For gas lasers,

$$p(v_a) = \frac{2(\ln 2)^{1/2}}{\pi^{1/2} \Delta v_d} e^{-[2(v_a - v_o)/(\Delta v_d)]^2 \ln 2} \quad (4.69)$$

due to the Doppler effect. Though Eq. (4.69) is exact for Doppler broadened gas lasers,

it is also a good approximation for most other types of lasers. Combining Eq. (4.69) with Eqs. (4.67) and (4.68) yields

$$n_{mixed}(x, y, z, t) = n_o(x, y, z, t) + \frac{cg_{ho}(x, y, z, t)}{4\pi v} \frac{2(\ln 2)^{1/2}}{\pi^{1/2} \Delta v_d} \times \int_0^\infty \frac{\left[\frac{2(v - v_a)}{\Delta v_h} \right] e^{-[2(v_a - v_o)/(\Delta v_d)]^2 \ln 2} dv_a}{1 + \left[\frac{2(v - v_a)}{\Delta v_h} \right]^2 + sI(x, y, z, t)}, \quad (4.70)$$

$$g_{mixed}(x, y, z, t) = g_{ho}(x, y, z, t) \frac{2(\ln 2)^{1/2}}{\pi^{1/2} \Delta v_d} \int_0^\infty \frac{e^{-[2(v_a - v_o)/(\Delta v_d)]^2 \ln 2} dv_a}{1 + \left[\frac{2(v - v_a)}{\Delta v_h} \right]^2 + sI(x, y, z, t)}. \quad (4.71)$$

Inhomogeneous Broadening

When all of the laser atoms do not have stimulated transitions with the same energy level lifetime and energy difference, the laser is inhomogeneously broadened. In gas lasers, the laser atoms travel with different velocities which causes Doppler broadening. Inhomogeneous broadening also occurs due to frequency shifts caused by electric fields (Stark broadening) and magnetic fields (Zeeman broadening). An effective mass effect, important for small atoms, and a volume effect, important for large atoms, give rise to isotope shifts which are also a source of additional broadening. The atom's hyperfine structure may also encourage inhomogeneous broadening.

It is difficult to derive a general expression for the inhomogeneous linewidth of a laser. However, for Doppler broadening, the inhomogeneous Doppler linewidth is

$$\Delta v_d = 2v_o \left[\frac{2kT}{Mc^2} \ln 2 \right]^{1/2}. \quad (4.72)$$

For Doppler broadened lasers the refractive index and gain are

$$n_{doppler}(x, y, z, t) = n_o(x, y, z, t) + \frac{cg_{do}(x, y, z, t)}{2\pi^{3/2} v} F(\hat{x}) \quad (4.73)$$

$$g_{doppler}(x, y, z, t) = \frac{g_{do}(x, y, z, t) e^{-\hat{x}^2}}{[1 + sI(x, y, z, t)]^{1/2}} \quad (4.74)$$

where

$$\hat{x} \equiv 2(\ln 2)^{1/2}(\nu - \nu_o)/\Delta\nu_d \quad (4.75)$$

$$F(\hat{x}) \equiv e^{-\hat{x}^2} \int_0^{\hat{x}} e^{t^2} dt \quad (4.76)$$

The function $F(\hat{x})$ is known as Dawson's integral. At line center, the value of Dawson's integral is zero. It appears that the gain saturates more slowly for a Doppler broadened amplifier [Eq. (4.74)] than for a homogeneously broadened amplifier [Eq. (4.64)]. This effect exists because a Doppler broadened medium is made of many homogeneous frequency classes. As one saturates, there are other unsaturated frequency classes that provide additional gain.

Homogeneous and Doppler broadening are two idealized limits of the integral Eqs. (4.70) and (4.71). However, most lasers can generally be categorized as either homogeneously broadened or inhomogeneously broadened. The broadening mechanism that dominates depends upon the operating conditions. For example, CO_2 lasers may be changed from homogeneously broadened to Doppler broadened by changing the laser amplifier temperature or pressure. To determine the dominant broadening mechanism, one compares the homogeneous linewidth with the Doppler linewidth. If one is several times the other, then that broadening mechanism dominates. At high pressure, collision broadening dominates and CO_2 is homogeneously broadened. Similarly, at low pressure, the mean time between collisions is larger, and the laser becomes Doppler broadened. The homogeneous linewidth is linear in temperature, but the Doppler linewidth is proportional to the square root of temperature. If the operating temperature is small enough, even some solid state lasers become Doppler broadened.

Near-Doppler Broadening

For lasers that cannot be classified as either homogeneously broadened or inhomogeneously broadened, one must integrate Eqs. (4.67) and (4.68) or (4.70) and (4.71) directly. In general, numerical methods are used. However, if only the first-order effects of the homogeneous linewidth on a Doppler broadened laser are considered, then the refractive index and gain for a nearly Doppler broadened laser are

$$n_{near\ doppler}(x, y, z, t) = n_o(x, y, z, t) \left\{ 1 + \frac{c g_{do}(x, y, z, t)}{\pi^{3/2} v n_o(x, y, z, t)} \left[F(\hat{x}) - \hat{\epsilon} \pi^{1/2} [1 + sI(x, y, z, t)]^{1/2} \hat{x} e^{-\hat{x}^2} \right] \right\}^{1/2} \quad (4.77)$$

$$g_{near\ doppler}(x, y, z, t) = g_{do}(x, y, z, t) \times \left[\frac{e^{-\hat{x}^2}}{[1 + sI(x, y, z, t)]^{1/2}} + 2\pi^{-1/2} \hat{\epsilon} (e^{-\hat{x}^2} + 2\hat{x}F(\hat{x}) - 1) \right] \quad (4.78)$$

where

$$\hat{\epsilon} \equiv (\ln 2)^{1/2} \frac{\Delta v_h}{\Delta v_d} . \quad (4.79)$$

While the previous refractive index expression for a homogeneous medium is only valid for low gain media, Eq. (4.77) does not have this restriction.

SUMMARY

With few constraints and approximations, we have reduced Maxwell's Equations to a single wave equation for the electric field in a saturating medium such as a laser amplifier. Gain and dispersion characteristics of several different types of laser media have been reviewed. With the results of this chapter, several different matrix theories may be derived.

CHAPTER V

LASER BEAMS IN OPTICAL SYSTEMS

INTRODUCTION

A complex optical system is a sequence of optical elements in which one or more of the elements provides gain or loss to an input light beam. Gaussian transmission filters (and Gaussian variable reflectivity mirrors) [22] [153] [155]-[159], exponential transmission filters (and exponential variable reflectivity mirrors) [138], homogeneous amplifiers and absorbers [134], amplifiers (and absorbers) with a linear gain (loss) profile [17] [138], and amplifiers (and absorbers) with a quadratic gain (loss) profile [1] [14] [145] are all complex optical elements. Amplifiers and absorbers may be in the shape of a wedge or lens [138]. A medium which may have both a quadratic gain and refractive index profile is known as a complex lenslike medium [1] while a similar linearly profiled medium is a complex prismlike medium [138].

Complex optical elements are often used in modern optical design. Gaussian variable reflectivity mirrors have been used in CO_2 [214]-[216] and $Nd:YAG$ [218] -[222] lasers for LIDAR [223] and high power applications. High diffraction loss ("unstable resonator") lasers are very stable against temperature fluctuations, external vibrations and other mechanical perturbations [222], and mode-media instabilities [224]. Resonators with Gaussian variable reflectivity mirrors have several of the advantages of conventional high diffraction loss resonators without suffering from poor mode quality. In the past, a disadvantage of variable reflectivity mirrors has been poor power handling characteristics. However, increased interest in the field has spawned several novel variable

reflectivity mirror designs which are simple and have a high damage threshold [225].

To some extent, every laser possesses some sort of gain profile and is thus a complex optical system. This effect is well documented for longitudinally pumped electron-impact-excited gas lasers which inevitably have a radial current distribution, and thus a radial gain profile [226]. Lasers which are optically pumped with a nonuniform field such as a Gaussian beam also possess nonuniform gain distributions [227]. Pump absorption in the laser amplifier also leads to an inhomogeneous gain profile. Gain profiles are also present in solid with nonuniform ion-doping and may exist in liquid lasers with inhomogeneous dye molarity. Many semiconductor lasers also possess transverse gain distributions. The subject of gain profiled media has been recently reviewed [228]. Though the existence of nonuniform transverse gain in laser amplifiers is often inevitable, the effect may sometimes be exploited in modern optical system design. For example, in high gain *Xe* and *He–Xe* lasers [1] and subsequently in double heterostructure *GaAs* lasers [229], the transverse gain profile has been used to simplify resonator design.

Lasers with Gaussian variable reflectivity mirrors and transverse gain profiles are only two examples of the plethora of complex Gaussian beam optical systems. Omnipresent misalignment effects in these systems have long been of interest as minute misalignments can have drastic effects on a laser's output [230]. Previously, numerical methods have been used to examine misalignment effects. Analytical techniques have also been of interest since it was found that a Gaussian beam is an eigenmode even of misaligned complex optical systems, and solutions were found using a complicated diffraction integral on an ad hoc basis [230] [231]. The analysis was later generalized to include higher-order modes [232].

Basically, ray matrix techniques may not be used to analyze these complex optical systems, and thus the purpose of this chapter is to generalize the Gaussian beam matrix method to include misaligned, complex optical systems.

Optical transfer matrix techniques are typically derived from Maxwell's equations. In the previous chapter, Maxwell's equations were reduced to wave Eq. (4.59) which with $\kappa(z) = k_o(z)$ is

$$\begin{aligned}
& \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} + \frac{\partial^2 \bar{\Psi}}{\partial z^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' - k'^2/4 + i \frac{dk_o}{dz} \right] \bar{\Psi} = \\
& - \nabla \left[\frac{[2(k_o - k'/2)\nabla(k_o - k'/2) + i\omega\mu\nabla\sigma] \cdot \bar{\Psi}}{k_o^2 - k_o k' + k'^2/4 + i\omega\mu\sigma} \right] \\
& + ik_o \left[\frac{[2(k_o - k'/2)\nabla(k_o - k'/2) + i\omega\mu\nabla\sigma] \cdot \bar{\Psi}}{k_o^2 - k_o k' + k'^2/4 + i\omega\mu\sigma} \right] \frac{1}{i_z} \\
& + \frac{2}{\omega} \left[-i(k_o - k'/2) \frac{\partial k'}{\partial t} + \frac{1}{4\omega} \left(\frac{\partial k'}{\partial t} \right)^2 - \frac{k_o - k'/2}{2\omega} \frac{\partial^2 k'}{\partial t^2} \right] \bar{\Psi} \\
& + \left[-\mu\sigma + \frac{2i(k_o^2 - k_o k' + k'^2/4)}{\omega} - \frac{2}{\omega^2} (k_o - k'/2) \frac{\partial k'}{\partial t} \right] \frac{\partial \bar{\Psi}}{\partial t} \\
& + \left[\frac{k_o^2 - k_o k' + k'^2/4 + i\omega\mu\sigma}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2} . \tag{5.1}
\end{aligned}$$

In the next section, approximations and constraints are used to reduce this wave equation to a simpler form.

APPROXIMATIONS OF THE WAVE EQUATION

The wave Eq. (5.1) is extremely general and applies to a wide range of situations encountered in practice. However, under typical operating conditions, this wave equation is unnecessarily complicated and it may be simplified substantially. The purpose of this section is use approximations and constraints to reduce the wave Eq. (5.1) to a simpler form while retaining the essential physics of laser beam propagation.

The vector $\bar{\Psi}$ contains three components which are coupled through Eq. (5.1). If

the medium in which the light beam propagates is homogeneous, then the vector coupling terms in Eq. (5.1) vanish. However, these coupling terms are often negligible even in inhomogeneous media. In particular, if

$$\left| -2\nabla \left[\frac{\nabla k \cdot \bar{\Psi}}{k} \right] \right| \ll |k^2 \bar{\Psi}| \quad (5.2)$$

then the scalar approximation may be used [49] [233] - [235]. Qualitatively, if the refractive index (or gain) varies negligibly in the distance of a wavelength of the optical signal, then Eq. (5.1) reduces to

$$\begin{aligned} \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} + \frac{\partial^2 \bar{\Psi}}{\partial z^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' - k'^2/4 + i \frac{dk_o}{dz} \right] \bar{\Psi} = \\ + \frac{2}{\omega} \left[-i(k_o - k'/2) \frac{\partial k'}{\partial t} + \frac{1}{4\omega} \left(\frac{\partial k'}{\partial t} \right)^2 - \frac{k_o - k'/2}{2\omega} \frac{\partial^2 k'}{\partial t^2} \right] \bar{\Psi} \\ + \left[-\mu\sigma + \frac{2i(k_o^2 - k_o k' + k'^2/4)}{\omega} - \frac{2}{\omega^2} (k_o - k'/2) \frac{\partial k'}{\partial t} \right] \frac{\partial \bar{\Psi}}{\partial t} \\ + \left[\frac{k_o^2 - k_o k' + k'^2/4 + i\omega\mu\sigma}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2} . \end{aligned} \quad (5.3)$$

The scalar approximation has greatest validity for the two dominant transverse components [14]. However, the longitudinal component of the electric field can be obtained from, for example, Gauss's law for the electric field. Similarly, the magnetic field can be obtained from the electric field using Faraday's law. It should be emphasized that if the complex propagation "constant" is really constant (i.e., space independent), then the coupling term is identically zero, and no approximation has been made. In either case the equations become uncoupled and can be solved separately. In this scalar approximation, it is therefore possible to define a propagation constant that is different for each of the two rectangular field components, and this is a common way to account for media anisotropy [161].

One of the more useful properties of laser light is its small beam divergence, and most laser designs produce low divergence light. For such light beams it is appropriate to make the paraxial approximation [154] [236]:

$$\left| \frac{\partial}{\partial z} \left(\frac{\partial \bar{\Psi}}{\partial z} \right) \right| \ll \left| 2ik_o \frac{\partial \bar{\Psi}}{\partial z} \right|. \quad (5.4)$$

The paraxial approximation is distinct, though consistent with the scalar approximation. In particular, the scalar approximation is valid when the material properties do not vary in the distance of a wavelength, while the paraxial approximation is valid when the derivative of the complex amplitude of the electric field does not have any significant variation in the distance of a wavelength. With the paraxial approximation, the wave equation is reduced to

$$\begin{aligned} \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' - k'^2/4 + i \frac{dk_o}{dz} \right] \bar{\Psi} = \\ + \frac{2}{\omega} \left[-i(k_o - k'/2) \frac{\partial k'}{\partial t} + \frac{1}{4\omega} \left(\frac{\partial k'}{\partial t} \right)^2 - \frac{k_o - k'/2}{2\omega} \frac{\partial^2 k'}{\partial t^2} \right] \bar{\Psi} \\ + \left[-\mu\sigma + \frac{2i(k_o^2 - k_o k' + k'^2/4)}{\omega} - \frac{2}{\omega^2} (k_o - k'/2) \frac{\partial k'}{\partial t} \right] \frac{\partial \bar{\Psi}}{\partial t} \\ + \left[\frac{k_o^2 - k_o k' + k'^2/4 + i\omega\mu\sigma}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2}. \end{aligned} \quad (5.5)$$

Under certain conditions, the paraxial approximation may be avoidable [237] [238].

Just as we do not expect the material properties to vary in the distance of a wavelength (so we use the scalar approximation), we also do not expect the material properties to vary in the time of an optical cycle. If this is the case,

$$\left| \frac{\partial}{\partial t} \left(\frac{\partial k'}{\partial t} \right) \right| \ll \left| 4\omega \frac{\partial k'}{\partial t} \right| \quad (5.6)$$

and the wave Eq. (5.5) reduces to

$$\begin{aligned}
& \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' - k'^2/4 + i \frac{dk_o}{dz} \right] \bar{\Psi} = \\
& \frac{2}{\omega} \left[-i(k_o - k'/2) \frac{\partial k'}{\partial t} + \frac{1}{4\omega} \left(\frac{\partial k'}{\partial t} \right)^2 \right] \bar{\Psi} \\
& + \left[-\mu\sigma + \frac{2i(k_o^2 - k_o k' + k'^2/4)}{\omega} - \frac{2}{\omega^2} (k_o - k'/2) \frac{\partial k'}{\partial t} \right] \frac{\partial \bar{\Psi}}{\partial t} \\
& + \left[\frac{k_o^2 - k_o k' + k'^2/4 + i\omega\mu\sigma}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2} . \tag{5.7}
\end{aligned}$$

Similar to the previous approximation, in all but the most extreme conditions,

$$\left| \frac{\partial k'}{\partial t} \right| \ll |\omega k_o| . \tag{5.8}$$

and the wave equation reduces further:

$$\begin{aligned}
& \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' - k'^2/4 + i \frac{dk_o}{dz} \right] \bar{\Psi} = \frac{2}{\omega} \left[-ik_o \frac{\partial k'}{\partial t} \right] \bar{\Psi} \\
& + \left[-\mu\sigma + \frac{2i(k_o^2 - k_o k' + k'^2/4)}{\omega} \right] \frac{\partial \bar{\Psi}}{\partial t} + \left[\frac{k_o^2 - k_o k' + k'^2/4 + i\omega\mu\sigma}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2} . \tag{5.9}
\end{aligned}$$

In optical system design, the conductivity of any individual component is small so that the skin depth $[\delta = (\pi f \mu \sigma)^{-1/2}]$ is much larger than a wavelength. In this case

$$\sigma \ll \left| \frac{k_o^2}{\mu \omega} \right| \tag{5.10}$$

and the wave equation becomes

$$\begin{aligned}
& \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' - k'^2/4 + i \frac{dk_o}{dz} \right] \bar{\Psi} = \frac{2}{\omega} \left[-ik_o \frac{\partial k'}{\partial t} \right] \bar{\Psi} \\
& + \left[\frac{2i(k_o^2 - k_o k' + k'^2/4)}{\omega} \right] \frac{\partial \bar{\Psi}}{\partial t} + \left[\frac{k_o^2 - k_o k' + k'^2/4}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2} . \tag{5.11}
\end{aligned}$$

Consistent with the paraxial approximation and the scalar approximation we consider only media with slow temporal and transverse spatial variations, so that

$$|k'^2/4| \ll |k_o^2| . \quad (5.12)$$

The wave equation reduces to

$$\begin{aligned} \frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' + i \frac{dk_o}{dz} \right] \bar{\Psi} = \frac{2}{\omega} \left[-ik_o \frac{\partial k'}{\partial t} \right] \bar{\Psi} \\ + \left[\frac{2i(k_o^2 - k_o k')}{\omega} \right] \frac{\partial \bar{\Psi}}{\partial t} + \left[\frac{k_o^2 - k_o k'}{\omega^2} \right] \frac{\partial^2 \bar{\Psi}}{\partial t^2} . \end{aligned} \quad (5.13)$$

The temporal analogy to the paraxial approximation is

$$\left| \frac{\partial}{\partial t} \left[\frac{\partial \bar{\Psi}}{\partial t} \right] \right| \ll \left| 2i \omega \frac{\partial \bar{\Psi}}{\partial t} \right| . \quad (5.14)$$

With this approximation $d\bar{\Psi}/dt$ is not allowed to vary on the time scale of an optical cycle. The wave Eq. (5.13) becomes

$$\frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial y^2} - 2ik_o \frac{\partial \bar{\Psi}}{\partial z} - \left[k_o k' + i \frac{dk_o}{dz} \right] \bar{\Psi} = \frac{2}{\omega} \left[-ik_o \frac{\partial k'}{\partial t} \right] \bar{\Psi} + \left[\frac{2i(k_o^2 - k_o k')}{\omega} \right] \frac{\partial \bar{\Psi}}{\partial t} . \quad (5.15)$$

Since we have made the scalar approximation, the wave equation has become three uncoupled scalar equations, and the substitution

$$\bar{\Psi}(x, y, z, t) \equiv \psi(x, y, z, t) [(\cos \zeta) \bar{i}_x + (\sin \zeta) \bar{i}_y] \quad (5.16)$$

reduces the vector wave equation to a scalar equation. In this case, the scalar wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - 2ik_o \frac{\partial \psi}{\partial z} - \left[k_o k' + i \frac{dk_o}{dz} \right] \psi = \frac{2}{\omega} \left[-ik_o \frac{\partial k'}{\partial t} \right] \psi + \left[\frac{2i(k_o^2 - k_o k')}{\omega} \right] \frac{\partial \psi}{\partial t} . \quad (5.17)$$

Similar, but more drastic than the approximation in Eq. (5.6), if ψ does not vary in an optical cycle, then the wave equation becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - 2ik_o \left[\frac{\partial \psi}{\partial z} + \frac{k_o}{\omega} \frac{\partial \psi}{\partial t} \right] - \left[k_o k' + i \frac{dk_o}{dz} - \frac{2i k_o}{\omega} \frac{\partial k'}{\partial t} \right] \psi = 0 . \quad (5.18)$$

There are two different methods employed in understanding pulsed laser beam behavior - the frequency domain method and the time domain method. Usually, dispersion characteristics of optical materials is specified in the frequency domain. Thus, time domain analysis requires that a Fourier transform of the wave equation, specification of the gain and index of the medium in terms of frequency, inverse Fourier transforming the equation, and finally solving the resulting equation. To simplify this process, the propagation constants in the frequency-dependent Fourier transformed equation are often series expanded in frequency, so that the resulting inverse transformed equation is only second-order in time.

For the following discussion, only nondispersive nonsaturating media are considered. Even for this restricted case, for arbitrary variations of the complex propagation constant, it is difficult to obtain exact analytic solutions of the paraxial wave Eq. (5.18). However, analytic solutions of Eq. (5.18) can be found under a wide variety of conditions if the propagation constant is Taylor series expanded in the transverse directions keeping only constant, linear, and quadratic terms:

$$k'(x, y, z, t) = k_{2x}(z)x^2 + k_{2y}(z)y^2 + k_{2t}(z)t^2 + k_{xy}(z)xy + k_{xt}(z)xt + k_{yt}(z)yt + k_{1x}(z)x + k_{1y}(z)y + k_{1t}(z)t \quad (5.19)$$

The idea of a series expansion is consistent with the requirement of slow temporal and transverse spatial variations given in approximation Eq. (5.12).

With these approximations and constraints, the solution of the wave Eq. (5.18) with the propagation constant given in Eq. (5.19) is

$$\psi(x, y, z, t) = \psi'(x, y, z, t) \exp \{ -i [Q_{2x}(z)x^2/2 + Q_{2y}(z)y^2/2 + Q_{2t}(z)t^2/2 + Q_{xy}(z)xy + Q_{xt}(z)xt + Q_{yt}(z)yt + S_{1x}(z)x + S_{1y}(z)y + S_{1t}(z)t + P(z)] \} \quad (5.20)$$

where the z-dependent parameters are chosen so that

$$Q_x^2 + Q_{xy}^2 + k_o \frac{\partial Q_x}{\partial z} + k_o k_{2x} = 0 \quad (5.21)$$

$$Q_y^2 + Q_{xy}^2 + k_o \frac{\partial Q_y}{\partial z} + k_o k_{2y} = 0 \quad (5.22)$$

$$Q_{xt}^2 + Q_{yt}^2 + k_o \frac{\partial Q_t}{\partial z} + k_o k_{2t} = 0 \quad (5.23)$$

$$2(Q_x + Q_y)Q_{xy} + 2k_o \frac{\partial Q_{xy}}{\partial z} + k_o k_{xy} = 0 \quad (5.24)$$

$$2Q_x Q_{xt} + 2Q_{yt} Q_{xy} + 2k_o \frac{\partial Q_{xt}}{\partial z} + k_o k_{xt} = 0 \quad (5.25)$$

$$2Q_y Q_{yt} + 2Q_{xt} Q_{xy} + 2k_o \frac{\partial Q_{yt}}{\partial z} + k_o k_{yt} = 0 \quad (5.26)$$

$$2Q_x S_x + 2Q_{xy} S_y + 2k_o \frac{\partial S_x}{\partial z} + 2\frac{k_o^2}{\omega} Q_{xt} + k_o k_{1x} - 2i\frac{k_o}{\omega} k_{xt} = 0 \quad (5.27)$$

$$2Q_y S_y + 2Q_{xy} S_x + 2k_o \frac{\partial S_y}{\partial z} + \frac{2k_o^2}{\omega} Q_{yt} + k_o k_{1y} - 2i\frac{k_o}{\omega} k_{yt} = 0 \quad (5.28)$$

$$2Q_{xt} S_x + 2Q_{yt} S_y + 2k_o \frac{\partial S_t}{\partial z} + \frac{2k_o^2}{\omega} Q_t + k_o k_{1t} - 4i\frac{k_o}{\omega} k_{2t} = 0 \quad (5.29)$$

$$2k_o \frac{\partial P}{\partial z} + S_x^2 + S_y^2 + \frac{2k_o^2}{\omega} S_t - 2i\frac{k_o}{\omega} k_{1t} + i(Q_x + Q_y) = 0 \quad (5.30)$$

and that $\psi'(x, y, z, t)$ is a constant for the fundamental mode considered here. Rather than consider specific solutions to these ordinary differential equations, our emphasis will be on the frequency domain method for solution of Eq. (5.18).

GAUSSIAN BEAMS IN CURVED COMPLEX LENS LIKE MEDIA

In this section, the frequency domain method is used to study Gaussian laser beams in curved complex lenslike media. In the frequency domain, the frequency dependence of the gain and refractive index of the medium in which the light beam propagates is assumed to be known. This frequency dependence was reported for several broad classes of lasers.

For frequency domain analyses, pure temporal sinusoids are examined individually. Thus, only optical media for which $k_{xt} = 0$ and $k_{yt} = 0$ are considered. For simplicity only, we also consider only those optical elements which do not twist so that $k_{xy} = 0$. A more general treatment that includes twisting lenslike media is given in Ref. 14. Under these conditions, the propagation constant has the form

$$k(x, y, z) = k_o(z) - [k_{2x}(z)x^2 + k_{2y}(z)y^2 + k_{1x}(z)x + k_{1y}(z)y]/2 . \quad (5.31)$$

With this total propagation constant, the continuous wave solutions to the wave Eq. (5.18) may be obtained with the substitution

$$\psi(x, y, z) = \psi'(x, y, z) \exp[-i[Q_x(z)x^2/2 + Q_y(z)y^2/2 + S_x(z)x + S_y(z)y]] \quad (5.32)$$

with the following definitions for the z-dependent parameters:

$$Q_x^2(z) + k_o(z) \frac{\partial Q_x(z)}{\partial z} + k_o(z)k_{2x}(z) = 0 \quad (5.33)$$

$$Q_y^2(z) + k_o(z) \frac{\partial Q_y(z)}{\partial z} + k_o(z)k_{2y}(z) = 0 \quad (5.34)$$

$$Q_x(z)S_x(z) + k_o(z) \frac{\partial S_x(z)}{\partial z} + \frac{k_o(z)k_{1x}(z)}{2} = 0 \quad (5.35)$$

$$Q_y(z)S_y(z) + k_o(z) \frac{\partial S_y(z)}{\partial z} + \frac{k_o(z)k_{1y}(z)}{2} = 0 . \quad (5.36)$$

The resulting differential equation is

$$\begin{aligned} \frac{\partial^2 \psi'}{\partial x^2} - 2i(S_x + Q_x x) \frac{\partial \psi'}{\partial x} + \frac{\partial^2 \psi'}{\partial y^2} - 2i(S_y + Q_y y) \frac{\partial \psi'}{\partial y} \\ - (S_x^2 + S_y^2) \psi' - i(Q_x + Q_y) \psi' - 2ik_o \frac{\partial \psi'}{\partial z} - i \frac{dk_o}{dz} \psi' = 0 . \end{aligned} \quad (5.37)$$

Equation (5.37) may be simplified substantially for fundamental mode propagation. In particular, if $\psi'(x, y, z) = \exp[-iP(z)]$ then the phase parameter equation becomes

$$2k_o(z) \frac{\partial P(z)}{\partial z} + S_x^2(z) + S_y^2(z) + i(Q_x(z) + Q_y(z)) = 0 , \quad (5.38)$$

an ordinary differential equation.

The complex propagation constant in Eq. (5.31) may be used to represent media that may have both a quadratic transverse refractive index profile and a quadratic transverse gain profile. The center of the both the gain profile and the index profile may be different and each may vary differently with axial distance. This may be seen by rewriting Eq. (5.31) as

$$\begin{aligned}
 k(x, y, z) = & \left\{ \beta_o + \frac{\beta_{1x}^2}{8\beta_{2x}} + \frac{\beta_{1y}^2}{8\beta_{2y}} - \frac{\beta_{2x}}{2} \left[x + \frac{\beta_{1x}}{2\beta_{2x}} \right]^2 - \frac{\beta_{2y}}{2} \left[y + \frac{\beta_{1y}}{2\beta_{2y}} \right]^2 \right\} \\
 & + i \left\{ \alpha_o + \frac{\alpha_{1x}^2}{8\alpha_{2x}} + \frac{\alpha_{1y}^2}{8\alpha_{2y}} - \frac{\alpha_{2x}}{2} \left[x + \frac{\alpha_{1x}}{2\alpha_{2x}} \right]^2 - \frac{\alpha_{2y}}{2} \left[y + \frac{\alpha_{1y}}{2\alpha_{2y}} \right]^2 \right\} \quad (5.39) \\
 = & \left\{ \left[\beta_o + \frac{\beta_{1x}^2}{8\beta_{2x}} + \frac{\beta_{1y}^2}{8\beta_{2y}} \right] - \frac{\beta_{2x}}{2} [x - d_{x\beta}]^2 - \frac{\beta_{2y}}{2} [y - d_{y\beta}]^2 \right\} \\
 & + i \left\{ \left[\alpha_o + \frac{\alpha_{1x}^2}{8\alpha_{2x}} + \frac{\alpha_{1y}^2}{8\alpha_{2y}} \right] - \frac{\alpha_{2x}}{2} [x - d_{x\alpha}]^2 - \frac{\alpha_{2y}}{2} [y - d_{y\alpha}]^2 \right\}, \quad (5.40)
 \end{aligned}$$

where each of the α and the β terms are in general z -dependent. The centers of the index profile $d_{x\beta}$ and $d_{y\beta}$, and the centers of the gain profile $d_{x\alpha}$ and $d_{y\alpha}$ are related to the gain and index coefficients by the relations

$$d_{x\beta}(z) = -\frac{\beta_{1x}(z)}{2\beta_{2x}(z)}, \quad (5.41)$$

$$d_{y\beta}(z) = -\frac{\beta_{1y}(z)}{2\beta_{2y}(z)}, \quad (5.42)$$

$$d_{x\alpha}(z) = -\frac{\alpha_{1x}(z)}{2\alpha_{2x}(z)}, \quad (5.43)$$

$$d_{y\alpha}(z) = -\frac{\alpha_{1y}(z)}{2\alpha_{2y}(z)}. \quad (5.44)$$

In a similar way, Eq. (5.32) may be rewritten as an off-axis Gaussian beam:

$$\begin{aligned} \psi(x, y, z) = \psi'(x, y, z) \exp & \left\{ \frac{Q_{xi}}{2} \left[x - \frac{S_{xi}}{Q_{xi}} \right]^2 + \frac{Q_{yi}}{2} \left[y - \frac{S_{yi}}{Q_{yi}} \right]^2 - \left[\frac{S_{xi}^2}{2Q_{xi}} + \frac{S_{yi}^2}{2Q_{yi}} \right] \right\} \\ & \times \exp \left\{ -i \left[\frac{Q_{xr}}{2} \left[x - \frac{S_{xr}}{Q_{xr}} \right]^2 + \frac{Q_{yr}}{2} \left[y - \frac{S_{yr}}{Q_{yr}} \right]^2 - \left[\frac{S_{xr}^2}{2Q_{xr}} + \frac{S_{yr}^2}{2Q_{yr}} \right] \right] \right\} \quad (5.45) \end{aligned}$$

$$\begin{aligned} & = \psi'(x, y, z) \exp \left\{ Q_{xi}(x - d_{xa})^2/2 + Q_{yi}(y - d_{ya})^2/2 - \left[\frac{S_{xi}^2}{2Q_{xi}} + \frac{S_{yi}^2}{2Q_{yi}} \right] \right\} \\ & \times \exp \left\{ -i \left[Q_{xr}(x - d_{xp})^2/2 + Q_{yr}(y - d_{yp})^2/2 - \left[\frac{S_{xr}^2}{2Q_{xr}} + \frac{S_{yr}^2}{2Q_{yr}} \right] \right] \right\} . \quad (5.46) \end{aligned}$$

Here, and everywhere after, the r and i subscripts are meant to denote the real and imaginary parts of a complex function, respectively. The center of the beam amplitude on the x -axis, d_{xa} , and the y -axis, d_{ya} , and the center of the beam phase on the x -axis, d_{xp} , and the y -axis, d_{yp} , are related to the beam parameters and the displacement parameters by the relations

$$d_{xa}(z) = -S_{xi}(z)/Q_{xi}(z) , \quad (5.47)$$

$$d_{xp}(z) = -S_{xr}(z)/Q_{xr}(z) . \quad (5.48)$$

Equations (5.33) and (5.34) are beam parameter equations, and Q_x and Q_y are known as beam parameters. Each of these two equations are uncoupled and can be solved independently. Since the forms of Eqs. (5.33) and (5.35) for the beam's parameters in the x -direction are identical to those of Eqs. (5.34) and (5.36), the beam's parameters in the y -direction, then for every equation written for the x -direction, there is a corresponding equation for the y -direction. Thus, though only x -direction equations are written, astigmatism is accounted for. The Gaussian beam's $1/e$ electric field amplitude spotsize and radius of phase front curvature at a flat plane are related to the beam parameter by the following relation:

$$Q_x(z) = \frac{\beta_o}{R_x(z)} - i \frac{2}{w_x^2(z)} . \quad (5.49)$$

The wavenumber $\beta_o = 2\pi n_o/\lambda$ where n_o is the refractive index of the medium, and λ is the wavelength of the laser beam.

Equation (5.35) is sometimes called the displacement parameter equation, and S_x is likewise known as the displacement parameter. Equation (5.35) is also uncoupled and can be solved independently once its corresponding beam parameter equation has been solved. The displacement parameter is related to the more physically pertinent beam amplitude center and the beam phase center through Eqs. (5.47) and (5.48). Equations (5.47) and (5.48) may be inverted and combined to yield

$$S_x(z) = -Q_{xr}(z)d_{xp}(z) - iQ_{xi}(z)d_{xa}(z) . \quad (5.50)$$

As the displacement of the amplitude center represents the position of a Gaussian beam, the slope of a Gaussian beam can be obtained by taking a z-derivative of the beam position given in Eq. (5.47) and substituting Eqs. (5.33) and (5.35). In the x-direction the slope is found to be

$$\frac{dd_{xa}}{dz} = -\frac{\alpha_{2x}}{Q_{xi}}(d_x\alpha - d_{xa}) + \left[\frac{1 - \alpha_\lambda Q_{xr}/Q_{xi}}{1 + \alpha_\lambda^2} \right] (d_{xa} - d_{xp}) \frac{Q_{xr}}{\beta_o} , \quad (5.51)$$

$$= \frac{\alpha_{2x} w_x^2}{2} (d_x\alpha - d_{xa}) + \left[\frac{1 + \alpha_\lambda \pi n w_x^2 / (\lambda R_x)}{1 + \alpha_\lambda^2} \right] \left[\frac{d_{xa} - d_{xp}}{R_x} \right] , \quad (5.52)$$

where $\alpha_\lambda \equiv \alpha_o/\beta_o$, and the z-dependences have been suppressed for readability. Equation (5.52) is written in this form to reduce to Eq. 14 of Ref. 17 when $\alpha_{2x} = 0$ and $\alpha_\lambda = 0$. In this way, the beam slope is related to the phase center of a Gaussian beam. Equation (5.50) can therefore be rewritten

$$S_x = -Q_{xr}d_{xa} + \beta_o \frac{1 + \alpha_\lambda^2}{1 - \alpha_\lambda Q_{xr}/Q_{xi}} \left[d'_{xa} + \frac{\alpha_{2x}}{Q_{xi}} (d_x\alpha - d_{xa}) \right] - iQ_{xi}d_{xa} \quad (5.53)$$

where $d'_{xa} \equiv dd_{xa}/dz$. Often, the input and output planes are chosen to be in a homogene-

ous dielectric medium (or freespace), and at these locations Eq. (5.53) reduces to

$$S_x = -Q_x d_{xa} + \beta_o d'_{xa} . \quad (5.54)$$

The goal now is to solve the above equations Eqs. (5.33) - (5.36) and express the results in terms of beam matrices. This is achieved by making the Ricatti substitution,

$$\frac{Q_x(z)}{k_o(z)} = \frac{1}{u_x(z)} \frac{du_x}{dz} , \quad (5.55)$$

into Eqs. (5.33) and (5.35) which may now be expressed as

$$\frac{d}{dz} \left[k_o(z) \frac{du_x}{dz} \right] + k_{2x}(z) u_x(z) = 0 , \quad (5.56)$$

$$\frac{d}{dz} \left[u_x(z) S_x(z) \right] = -\frac{1}{2} k_{1x}(z) u_x(z) . \quad (5.57)$$

As a linear homogeneous second order differential equation, Eq. (5.56) has solutions that may be written as a linear combination of two independent functions:

$$u_x(z) = A_x(z) u_x(0) + B_x(z) u'_x(0) . \quad (5.58)$$

The derivative may be written in a similar form:

$$u'_x(z) = C_x(z) u_x(0) + D_x(z) u'_x(0) . \quad (5.59)$$

By taking the z-derivative of Eq. (5.58), it follows that in material media $C_x(z) = dA_x/dz$ and $D_x(z) = dB_x/dz$. By substituting Eqs. (5.58) and (5.59) into Eq. (5.56), and using the fact that $A_x(z)$ and $B_x(z)$ are linearly independent, it follows that

$$\frac{d}{dz} [k_o(z) C_x(z)] = -k_{2x}(z) A_x(z) , \quad (5.60)$$

$$\frac{d}{dz} [k_o(z) D_x(z)] = -k_{2x}(z) B_x(z) . \quad (5.61)$$

These results may be used, for example, to show that the z-derivative of the product of $k_o(z)$ and the Wronskian, $A_x(z) D_x(z) - B_x(z) C_x(z)$, is zero, from which it follows that

$$A_x(z) D_x(z) - B_x(z) C_x(z) = \frac{k_o(0)}{k_o(z)} , \quad (5.62)$$

where $k_o(0)$ and $k_o(z)$ are the axial complex propagation constants at the input plane and output plane, respectively.

Equation (5.57) may be integrated directly, and using Eq. (5.58) it can be seen that

$$u_x(z)S_x(z) = u_x(0)S_x(0) - \left[\frac{1}{2} \int_0^z k_{1x}(z') A_x(z') dz' \right] u_x(0) - \left[\frac{1}{2} \int_0^z k_{1x}(z') B_x(z') dz' \right] u_x'(0). \quad (5.63)$$

If we define

$$G(z) \equiv -\frac{1}{2} \int_0^z k_{1x}(z') A_x(z') dz' , \quad (5.64)$$

$$H(z) \equiv -\frac{1}{2} \int_0^z k_{1x}(z') B_x(z') dz' , \quad (5.65)$$

then Eqs. (5.58), (5.59), and (5.63) can be written in a new "Generalized Beam Matrix" form:

$$\begin{bmatrix} u_x(z) \\ [Q_x(z)/k_o(z)]u_x(z) \\ S_x(z)u_x(z) \end{bmatrix} = \begin{bmatrix} A_x(z) & B_x(z) & 0 \\ C_x(z) & D_x(z) & 0 \\ G_x(z) & H_x(z) & 1 \end{bmatrix} \begin{bmatrix} u_x(0) \\ [Q_x(0)/k_o(0)]u_x(0) \\ S_x(0)u_x(0) \end{bmatrix} , \quad (5.66)$$

where Eq. (5.55) has also been used. In SI units, the dimension of $B_x(z)$ is meters, the dimensions of $C_x(z)$ and $G_x(z)$ are per meter, and $A_x(z)$, $D_x(z)$, and $H_x(z)$ are dimensionless.

The three rows of the matrix Eq. (5.66) represent three equations. It follows by dividing the second equation by the first that

$$\frac{Q_x(z)}{k_o(z)} = \frac{C_x(z) + D_x(z)Q_x(0)/k_o(0)}{A_x(z) + B_x(z)Q_x(0)/k_o(0)} . \quad (5.67)$$

This is the Kogelnik Transformation [144]. The beam parameter Q_x is related to the light beam's spotsize and radius of curvature through Eq. (5.49). In a similar manner, the third equation divided by the first equation in Eq. (5.66) yields a new law analogous to the Kogelnik transformation:

$$S_x(z) = \frac{S_x(0)}{A_x(z) + B_x(z)Q_x(0)/k_o(0)} + \frac{G_x(z) + H_x(z)Q_x(0)/k_o(0)}{A_x(z) + B_x(z)Q_x(0)/k_o(0)} . \quad (5.68)$$

For aligned optical systems, $G_x = 0$ and $H_x = 0$ and Eq. (5.68) reduces to a transformation derived previously [16]. In the appendix of Ref. 138 it is shown that when the generalized beam matrix elements are purely real (i.e. for lossless optical systems), the propagation characteristics may be described in terms of a generalized ray matrix. Once the generalized beam matrix is known, the procedure to propagate a Gaussian beam's position, slope, spotsize, and phase front curvature involves the use of the intermediate complex parameters Q_x and S_x which may be obtained from Eqs. (5.49) and (5.53), respectively. The output beam parameter, $Q_x(z)$, and the output displacement parameter, $S_x(z)$, may be obtained from Eqs. (5.67) and (5.68). The Gaussian beam's output parameters of interest may then be obtained from Eqs. (5.49) and (5.53).

As a specific example of a generalized beam matrix, a complex lenslike medium where $k_o(z) = k_o$ and $k_{2x}(z) = k_{2x}$ is considered. In this case, Eq. (5.56) is easily solved, and the corresponding generalized beam matrix is

$$T_{\text{complex lenslike medium}} = \begin{bmatrix} \cos(\gamma_x z) & \gamma_x^{-1} \sin(\gamma_x z) & 0 \\ -\gamma_x \sin(\gamma_x z) & \cos(\gamma_x z) & 0 \\ -\frac{1}{2} \int_0^z k_{1x}(z') \cos(\gamma_x z') dz' & -\frac{1}{2} \gamma_x^{-1} \int_0^z k_{1x}(z') \sin(\gamma_x z') dz' & 1 \end{bmatrix}, \quad (5.69)$$

where Eqs. (5.64) and (5.65) have been used as has the definition

$$\gamma_x \equiv \left[\frac{k_{2x}}{k_o} \right]^{1/2}. \quad (5.70)$$

For z -independent k_o and k_{2x} , the $G_x(z)$ and $H_x(z)$ integrations in Eq. (5.69) may be obtained for arbitrarily curved media by Taylor series expanding $k_{1x}(z)$ and applying successive integration by parts and noting Eqs. (5.60), (5.61). Though this procedure to propagate an off-axis Gaussian beam in a curved complex lenslike medium is straightfor-

ward, the usefulness of the generalized beam matrices is amplified by its application to more complicated systems of optical elements, which is considered in the next section.

GAUSSIAN BEAMS IN OPTICAL SYSTEMS

In the previous section, generalized beam matrices were introduced and the generalized beam matrix for a curved complex lenslike medium was identified. The purpose of this section is to identify generalized beam matrices for several other optical elements and apply the matrix method to optical systems. To encourage analogies, optical elements are divided into three classes: non-profiled, linearly-profiled, and quadratically-profiled. Non-profiled optical elements such as homogeneous media possess no phase variation or amplitude transmission variation in either of the transverse directions. Similarly, linearly-profiled optical elements are linear in the transverse variations of their complex propagation constants. Prisms and tilted mirrors are familiar examples of linearly-profiled optics. In the same manner, lenses and spherical mirrors (which are approximately parabolic) are examples of quadratically-profiled optical elements.

For a system of optical elements, it is sensible to designate the various reference planes numerically. In particular, the generalized beam matrix may be written

$$\begin{bmatrix} u_{x2} \\ (Q_{x2}/k_{o2})u_{x2} \\ S_{x2}u_{x2} \end{bmatrix} = \begin{bmatrix} A_x & B_x & 0 \\ C_x & D_x & 0 \\ G_x & H_x & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ (Q_{x1}/k_{o1})u_{x1} \\ S_{x1}u_{x1} \end{bmatrix}, \quad (5.71)$$

where the "1" and "2" subscripts represent the input and output parameters of the optical element or system, respectively. As solutions to the differential equations discussed in the previous section require, the bottom, rightmost element of the generalized beam matrix is unity. However, since the observable beam properties exist only as ratios of u_x , $Q_x k_o^{-1} u_x$, and $S_x u_x$, these observables are unchanged if the beam matrix is multiplied by any nonzero z-dependent function. Thus, if the bottom, rightmost element is initially nonzero, the generalized beam matrix may be scaled to make it unity. However, there

may exist certain peculiar matrices where the bottom, rightmost element is zero. In this case the output position and slope of the Gaussian beam would be independent of the input position and slope. Though it is straightforward to extend the results below to account for this effect, this possibility is excluded from further consideration here.

As before, the Kogelnik transformation is obtained by dividing the second row of Eq. (5.71) by the first row:

$$\frac{Q_{x2}}{k_{o2}} = \frac{C_x + D_x Q_{x1}/k_{o1}}{A_x + B_x Q_{x1}/k_{o1}} . \quad (5.72)$$

The displacement transformation is obtained by dividing the third row of Eq. (5.71) by the first row:

$$S_{x2} = \frac{S_{x1}}{A_x + B_x Q_{x1}/k_{o1}} + \frac{G_x + H_x Q_{x1}/k_{o1}}{A_x + B_x Q_{x1}/k_{o1}} . \quad (5.73)$$

An alternate, but redundant transformation may be obtained either by dividing the second row of Eq. (5.71) by the third row, or by dividing Eq. (5.73) by Eq. (5.72).

As is typical in transfer matrix methods, system analysis merely consists of matrix multiplication of the individual elements in reverse order. Though the procedure to obtain the generalized beam matrix for any medium described by Eq. (5.31) has been shown above, the only generalized beam matrix derived thus far represents a complex lenslike medium. Since the value of γ_x and the variation of k_{1x} in Eq. (5.69) are arbitrary, it is straightforward to analyze optical systems which consist only of piecewise continuous curved complex lenslike media with these different properties. In the limit of small γ_x , an arbitrarily curved complex lenslike medium is a special case of a complex prismlike medium. For arbitrary variations of $k_o(z)$ and $k_1(z)$, Eqs. (5.56), (5.64), and (5.65) are simply integrals since $k_{2x}(z) = 0$, and the matrix for a complex prismlike medium is

$$T_{\text{complex prismlike medium}} = \begin{bmatrix} 1 & \int_0^d k_o(0)k_o^{-1}(z)dz & 0 \\ 0 & k_o(0)k_o^{-1}(z) & 0 \\ -\frac{1}{2}\int_0^d k_{1x}(z)dz & -\frac{1}{2}\int_0^d k_{1x}(z)\int_0^z k_o(0)k_o^{-1}(z')dz'dz & 1 \end{bmatrix}. \quad (5.74)$$

When $k_{1x}(z) = 0$ and k_o is constant, Eq. (5.74) this reduces to the generalized beam matrix for a homogeneous medium.

Axis Transformation

The matrix representations for several nonprofiled optical elements are well known, and are listed in the first two sections of Table VI. However, the boundary matrix and axial medium matrix are generalized here so that the low nonsaturating gain per wavelength approximation is not made [134]. Small tilts of an optical element or system of elements may be accounted for by the use of a z-axis transformation (see for example [239]). Thus, the purpose of this subsection is to derive the generalized beam matrix for two optical elements. The first "axis transformation" changes the position and slope of the optic axis. The second beam matrix is for an aligned optical system which is tilted with respect to the optic axis.

TABLE VI
GENERALIZED BEAM MATRICES FOR NON-PROFILED ELEMENTS

Optical Medium	Generalized Beam Matrix Tangential Plane (x)	Generalized Beam Matrix Sagittal Plane (y)
Homogeneous Medium	$\begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Aligned Mirror	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Aligned Boundary	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k_o/k_{o2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k_o/k_{o2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Retro-reflector	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Axial Medium	$\begin{bmatrix} 1 & k_o(0) \int_0^d k_o^{-1}(z) dz & 0 \\ 0 & k_o(0) k_o^{-1}(d) & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & k_o(0) \int_0^d k_o^{-1}(z) dz & 0 \\ 0 & k_o(0) k_o^{-1}(d) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Axis Transformation	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tan\theta_x & -x_o & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tan\theta_y & -y_o & 1 \end{bmatrix}$
Misaligned ABCD	$\begin{bmatrix} A_x & B_x & 0 \\ C_x & D_x & 0 \\ G_x & H_x & 1 \end{bmatrix}$	$\begin{bmatrix} A_y & B_y & 0 \\ C_y & D_y & 0 \\ G_y & H_y & 1 \end{bmatrix}$
$G \Rightarrow$	$\beta_o [(A_x - 1) \tan\theta_x - C_x (x_o + L \sin\theta_x)]$	$\beta_o [(A_y - 1) \tan\theta_y - C_y (y_o + L \sin\theta_y)]$
$H \Rightarrow$	$\beta_o [(1 - D_x) x_o + B_x \tan\theta_x - D_x L \sin\theta_x]$	$\beta_o [(1 - D_y) y_o + B_y \tan\theta_y - D_y L \sin\theta_y]$

The axis transformation matrix represents one of a class of thin optical elements. In optical systems in which the media at the input and the output are identical, many of these thin optical elements have the property that $A = D = 1$, $B = 0$. Therefore, for any such optical element, the displacement parameter transformation Eq. (5.73) reduces to

$$S_{x2} = S_{x1} + G_x + H_x \frac{Q_{x1}}{k_{o1}} , \quad (5.75)$$

and the Kogelnik transformation Eq. (5.72) reduces to

$$\frac{Q_{x2}}{k_{o2}} = C_x + \frac{Q_{x1}}{k_{o1}} . \quad (5.76)$$

To examine the effects of this transformation on the position and slope of a Gaussian beam, Eqs. (5.50) and (5.76) are substituted into Eq. (5.75) and the result is separated into real and imaginary parts. Without loss of generality, it is assumed that the media at the input and output of the system are freespace. It then follows from Eqs. (5.47) and (5.48) that

$$d_{xa2} = \frac{Q_{x1i} d_{xa1} - G_{xi} - H_{xi} Q_{x1r} \beta_o - H_{xr} Q_{x1i} \beta_o}{Q_{x1i} + \beta_o C_{xi}} , \quad (5.77)$$

$$d_{xp2} = \frac{Q_{x1r} d_{xp1} - G_{xr} - H_{xr} Q_{x1r} \beta_o + H_{xi} Q_{x1i} \beta_o}{Q_{x1r} + \beta_o C_{xr}} , \quad (5.78)$$

where, as before, the r and i subscripts represent the real and imaginary parts of the complex quantity, respectively. The phase and amplitude displacements are related to the beam slope from Eqs. (5.51). When the input and output media are freespace, Eqs. (5.51) reduce to

$$d'_{xa} = (d_{xa} - d_{xp}) \frac{Q_{xr}}{\beta_o} . \quad (5.79)$$

Combining Eqs. (5.76)-(5.79) yields

$$\begin{aligned} d'_{xa2} = & \left[\frac{C_{xr} Q_{x1i} - C_{xi} Q_{x1r}}{Q_{x1i} + \beta_o C_{xi}} \right] (d_{xa1} - H_{xr} \beta_o) + d'_{xa1} \\ & + \frac{G_{xr} - H_{xi} Q_{x1i} \beta_o}{\beta_o} - \left[\frac{Q_{x1r} + \beta_o C_{xr}}{Q_{x1i} + \beta_o C_{xi}} \right] \frac{G_{xi} + H_{xi} Q_{x1r} \beta_o}{\beta_o} . \end{aligned} \quad (5.80)$$

This equation used with Eq. (5.77) is useful for determining the effects of a thin element on the position and slope of a Gaussian beam. As an example, the Gaussian aperture is

considered. For an undisplaced aligned Gaussian aperture, $G_x = 0$, $H_x = 0$, and $C_x = -2i/\beta_o w_{ga,x}^2$ where $w_{ga,x}$ is the width of the aperture. In this case Eqs. (5.77) and (5.80) reduce to

$$d_{xa2} = \frac{d_{xa1}}{1 + (w_{x1}/w_{ga,x})^2} , \quad (5.81)$$

$$d'_{xa2} = -\frac{R_x^{-1} (w_{x1}/w_{ga,x})^2}{1 + (w_{x1}/w_{ga,x})^2} d_{xa1} + d'_{xa1} , \quad (5.82)$$

respectively. Equation (5.81) has been previously derived [22]. From Eqs. (5.81) and (5.82) it follows that a calibrated Gaussian aperture may be used to obtain a beam's phase front curvature and spotsize by merely measuring beam positions and slopes.

When $C_x = 0$, the Gaussian beam's spotsize and radius of curvature remain unchanged and Eqs. (5.77) and (5.80) take on simpler forms:

$$d_{xa2} = d_{xa1} - \frac{G_{xi}}{Q_{x1i}} - \frac{H_{xr}}{\beta_o} - \frac{Q_{x1r}}{Q_{x1i}} \frac{H_{xi}}{\beta_o} , \quad (5.83)$$

$$d'_{xa2} = d'_{xa1} + \frac{G_{xr}}{\beta_o} - \frac{Q_{x1r}}{Q_{x1i}} \frac{G_{xi}}{\beta_o} - \left[1 + \frac{Q_{x1r}^2}{Q_{x1i}^2} \right] \frac{Q_{x1i} H_{xi}}{\beta_o^2} . \quad (5.84)$$

If G_x is real and $H_x = 0$, then the slope is changed while the position remains unaltered. This slope change may occur due to the presence of an optical element, or due to a redefinition of the optic axis. Similarly, if H_x is real and $G_x = 0$, then the position is altered while the slope remains unchanged. These operations may be written

$$d_{xa2} = d_{xa1} + x_o , \quad (5.85)$$

$$d'_{xa2} = d'_{xa1} + \tan\theta_x . \quad (5.86)$$

where the displacement of the optical element, x_o , and the slope of the optical element, $\tan\theta_x$, are strictly real. These two operations may be combined into a single matrix using Eqs. (5.83) and (5.84) with real G_x and H_x , and the generalized beam matrix for this

optical element is

$$T_{axis \text{ transformation},x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta_o \tan \theta_x & -\beta_o x_o & 1 \end{bmatrix}. \quad (5.87)$$

Since the matrix elements are real, this generalized beam matrix may also be derived by using Eqs. (5.85) and (5.86) together with the generalized ray matrix conversion formulas Eqs. (A13) and (A14) in the appendix. However, it should be emphasized that the ray matrix techniques may not be used for complex-valued matrices, and could not be used, for example, to derive Eqs. (5.81) and (5.82).

The axis transformation matrix Eq. (5.87) may be used to obtain the generalized Gaussian beam matrix for a tilted optical system which would otherwise be representable by an ordinary complex ABCD matrix. The methodology consists of postmultiplying the ABCD matrix to tilt the optic axis, and premultiplying the ABCD matrix to transform the optic axis back to its original position and slope [239]. The difference of the position of the axis at the output from that at the input is $L \sin \theta_x$ where L is the length and θ_x is the tilt angle of the ABCD system. The matrix for an aligned system with a global tilt angle of θ is:

$$T_{tilted \text{ ABCD},x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta_o \tan \theta_x & -\beta_o (x_o + L \sin \theta_x) & 1 \end{bmatrix} \begin{bmatrix} A_x & B_x & 0 \\ C_x & D_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\beta_o \tan \theta_x & \beta_o x_o & 1 \end{bmatrix}, \quad (5.88)$$

$$= \begin{bmatrix} A_x & B_x & 0 \\ C_x & D_x & 0 \\ \beta_o [(A_x - 1) \tan \theta_x - C_x (x_o + L \sin \theta_x)] & \beta_o [B_x \tan \theta_x - D_x L \sin \theta_x + (1 - D_x) x_o] & 1 \end{bmatrix}. \quad (5.89)$$

This result along with the axis transformation matrix Eq. (5.89) are included in Table VI.

The above matrix allows for the analysis of paraxially tilted complex optical systems. For example, the Gaussian beam matrix for a spherical mirror with curvature R has $A_x = D_x = 1$, $B_x = 0$, $C_x = -2/R$, and $L = 0$. From Eq. (5.89), the generalized beam matrix is

$$T_{spherical\ mirror, x} = \begin{bmatrix} 1 & 0 & 0 \\ -2R^{-1} & 1 & 0 \\ 2\beta_o x_o R^{-1} & 0 & 1 \end{bmatrix}. \quad (5.90)$$

Linearly-Profiled Optical Elements

Generalized beam matrix representations for linearly-profiled optical elements are found in this section. While the thin prism and exponential aperture are optical elements linearly-profiled in complex phase, complex prismlike media are linearly-profiled in complex propagation constant.

The generalized Gaussian beam matrix for the complex prismlike medium introduced here is given by Eq. (5.74). The next optical element considered is an aperture which has an exponential transmission or reflection profile. Since the exponential aperture is a thin optical element, the output is just the input multiplied by the exponential transmission function. For an exponential of damping length $w_{ea, x}$, tilted at the small angle θ_x , and offset by the distance x_o , the output electric field is

$$\left\{ E_o' \exp[-i(Q_x x^2/2 + Q_y y^2/2 + S_x x + S_y y + P)] \right\}_{out} = \left\{ E_o' \exp[-i(Q_x x^2/2 + Q_y y^2/2 + S_x x + S_y y + P)] \right\}_{in} \exp[(x - x_o)/(w_{ea, x} \cos \theta_x)]. \quad (5.91)$$

The output beam parameter, displacement parameter, and phase parameter may be found in terms of the input conditions by equating terms in Eq. (5.91):

$$Q_{x2} = Q_{x1}, \quad (5.92)$$

$$S_{x2} = S_{x1} + \frac{i}{w_{ga,x} \cos \theta_x} , \quad (5.93)$$

$$P_2 = P_1 - \frac{ix_o}{w_{ga,x} \cos \theta_x} . \quad (5.94)$$

The generalized beam matrix for the exponential aperture for the x-direction may now be obtained from Eqs. (5.72) and (5.73):

$$T_{\text{exponential aperture},x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i/(w_{ea,x} \cos \theta_x) & 0 & 1 \end{bmatrix} . \quad (5.95)$$

By our postulate, the exponential aperture varied only in the x-direction, and thus the generalized beam matrix in the y-direction is the identity matrix. However, if the axis of the aperture was not parallel to the x-direction, there would be a similar matrix for the y-direction.

As an intermediate step to determining the generalized beam matrix for a thin prism, the matrix for a tilted linear boundary between two homogeneous media is found. The accumulated phase of a plane wave after propagating a distance $z > z'$ where z' is the position of a boundary between two media with propagation constants k_{o1} and k_{o2} is

$$E'_{out} = E'_{in} \exp[-ik_{o1}z'] \exp[-ik_{o2}(z-z')] , \quad (5.96)$$

$$= E'_{in} \exp[-ik_{o2}z] \exp[-i(k_{o1} - k_{o2})z'] . \quad (5.97)$$

Here, the boundary position is allowed to vary with transverse distance. The boundary under consideration is linear in the xz -plane and is defined as being displaced from the axis by an amount x_o . Thus,

$$z' = (x - x_o) \tan \theta_x , \quad (5.98)$$

where θ_x is measured from the x-axis (toward the z-axis). Equation (5.98) may be combined with Eq. (5.97), and it follows that

$$\begin{aligned}
\left\{ E_o' \exp[-i(Q_x x^2/2 + Q_y y^2/2 + S_x x + S_y y + P)] \right\}_{out} = \\
\left\{ E_o' \exp[-i(Q_x x^2/2 + Q_y y^2/2 + S_x x + S_y y + P)] \right\}_{in} \\
\times \exp[-ik_o z] \exp[-i(k_o1 - k_o2)(x - x_o) \tan \theta_x] . \quad (5.99)
\end{aligned}$$

Equating terms as before, the output phase parameter is

$$P_2 = P_1 - (k_o1 - k_o2)x_o \tan \theta_x \quad (5.100)$$

and the generalized beam matrix in the x-direction is

$$T_{linear \ boundary \ x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k_o1/k_o2 & 0 \\ (k_o1 - k_o2) \tan \theta_x & 0 & 1 \end{bmatrix} . \quad (5.101)$$

The matrix for a simple prism may be obtained by combining two linear boundaries:

$$T_{thin \ prism \ x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (k_o - \beta_{air})(\tan \theta_{x2} - \tan \theta_{x1}) & 0 & 1 \end{bmatrix} . \quad (5.102)$$

The thin prism is allowed to have gain or loss ($k_o = \beta_o + i\alpha_o$), as are several of the subsequent optical elements considered here. Thus, an amplifier (or absorber) wedge, for example, is included in Eq. (5.102).

The final optical element considered in this subsection is the tilted (flat) mirror. The tilted mirror has the same characteristic as Eq. (5.86), and thus the generalized beam matrix is given by Eq. (5.87) with $x_o = 0$. The results of this subsection are summarized in first half of Table VII.

TABLE VII
GENERALIZED BEAM MATRICES FOR PROFILED ELEMENTS

Optical Medium	Generalized Beam Matrix Tangential Plane (x)	Generalized Beam Matrix Sagittal Plane (y)
Prismlike Medium	$\begin{bmatrix} 1 & \int_0^d k_o(0)k_o^{-1}(z)dz & 0 \\ 0 & k_o(0)k_o^{-1}(d) & 0 \\ -\frac{1}{2}\int_0^d k_{1x}(z)dz & -\frac{1}{2}\int_0^d k_{1x}(z)\int_0^d k_o(0)k_o^{-1}(\zeta)d\zeta dz & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & \int_0^d k_o(0)k_o^{-1}(z)dz & 0 \\ 0 & k_o(0)k_o^{-1}(d) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Tilted Mirror	$\begin{bmatrix} 1 & 0 & 0 \\ \beta_o \tan \theta_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Tilted Boundary	$\begin{bmatrix} 1 & 0 & 0 \\ (k_{o1} - k_{o2})\tan \theta_x & k_{o1}/k_{o2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k_{o1}/k_{o2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Thin Prism	$\begin{bmatrix} 1 & 0 & 0 \\ (k_o - \beta_{air})(\tan \phi_{x2} - \tan \phi_{x1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Exponential Aperture	$\begin{bmatrix} 1 & 0 & 0 \\ i/(w_{ea,x} \cos \theta_x) & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ i/(w_{ea,y} \cos \theta_y) & 0 & 1 \end{bmatrix}$
Lenslike Medium	$\begin{bmatrix} \cos(\gamma_x d) & \gamma_x^{-1} \sin(\gamma_x d) & 0 \\ -\gamma_x \sin(\gamma_x d) & \cos(\gamma_x d) & 0 \\ -\frac{1}{2}\int_0^d k_{1x}(z) \cos(\gamma_x z) dz & -\frac{1}{2}\gamma_x^{-1} \int_0^d k_{1x}(z) \sin(\gamma_x z) dz & 1 \end{bmatrix}$	$\begin{bmatrix} \cos(\gamma_y d) & \gamma_y^{-1} \sin(\gamma_y d) & 0 \\ -\gamma_y \sin(\gamma_y d) & \cos(\gamma_y d) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Spherical Mirror	$\begin{bmatrix} -2R_x^{-1} \cos \theta_x & 0 & 0 \\ 2\beta_o x_o R_x^{-1} \cos \theta_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -2(R_y \cos \theta_x)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Spherical Boundary	$\begin{bmatrix} (1 - k_{o1}/k_{o2})R_x^{-1} \cos \theta_x & 0 & 0 \\ (k_{o1} - k_{o2})(\tan \theta_x + x_o R_x^{-1} \cos \theta_x) & k_{o1}/k_{o2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} (1 - k_{o1}/k_{o2})(R_y \cos \theta_x)^{-1} & 0 & 0 \\ 0 & k_{o1}/k_{o2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Thin Lens	$\begin{bmatrix} -f_{active,x}^{-1} \cos \theta_x & 0 & 0 \\ -\beta_{air} x_o f_{active,x}^{-1} \cos \theta_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -f_{active,y}^{-1} \cos \theta_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Gaussian Aperture	$\begin{bmatrix} -2i/(\beta_{air} w_{ga,x}^2 \cos^2 \theta_x) & 0 & 0 \\ 2ix_o/(w_{ga,x}^2 \cos^2 \theta_x) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -2i/(\beta_{air} w_{ga,y}^2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Quadratically-Profiled Optical Elements

When $S_x = 0$ the 3x3 matrix method reduces to the 2x2 method of Kogelnik. Thus, for elements in which $G_x = H_x = 0$, the A_x , B_x , C_x , and D_x elements are the same as those of conventional Gaussian beam optics. It is the purpose of this subsection to extend the range of validity of these conventional optical elements so that they may be displaced, misaligned, or curved.

The generalized beam matrix for a complex lenslike medium has been previously derived, and is given in Eq. (5.69). The next optical element considered is an aperture which has a Gaussian transmission or reflection profile. For a Gaussian aperture with $1/e$ amplitude width $w_{ga,x}$ tilted at an angle θ_x , and offset by the length x_o , the output electric field is

$$\left\{ E_o' \exp[-i(Q_x x^2/2 + Q_y y^2/2 + S_x x + S_y y + P)] \right\}_{out} =$$

$$\left\{ E_o' \exp[-i(Q_x x^2/2 + Q_y y^2/2 + S_x x + S_y y + P)] \right\}_{in}$$

$$\times \exp[-(x - x_o)^2/(w_{ga,x}^2 \cos^2 \theta_x)] \exp[-y^2/(w_{ga,y}^2)] . \quad (5.103)$$

As before, the output beam parameter, displacement parameter, and phase parameter may be found in terms of the input conditions by equating terms in Eq. (5.103):

$$\frac{Q_{x2}}{2} = \frac{Q_{x1}}{2} - \frac{i}{w_{ga,x}^2 \cos^2 \theta_x} , \quad (5.104)$$

$$S_{x2} = S_{x1} + i \frac{2x_o}{w_{ga,x}^2 \cos^2 \theta_x} , \quad (5.105)$$

$$P_2 = P_1 - i \frac{x_o^2}{w_{ga,x}^2 \cos^2 \theta_x} . \quad (5.106)$$

Without loss of generality, it may be assumed that the medium surrounding the aperture

is completely characterized by the relation $k(x, y, z) = \beta_{air}$. Under this condition, the generalized beam matrix may be obtained from Eqs. (5.72), (5.73), (5.104), and (5.105):

$$T_{Gaussian\ aperture, x} = \begin{bmatrix} 1 & 0 & 0 \\ -2i/(\beta_{air} w_{ga,x}^2 \cos^2 \theta_x) & 1 & 0 \\ 2ix_o/(w_{ga,x}^2 \cos^2 \theta_x) & 0 & 1 \end{bmatrix}. \quad (5.107)$$

There is a similar matrix governing the y -distribution of the fields.

Similar to the linear case, Eq. (5.97) may be used to obtain the matrix representation of spatially nonlinear boundaries between complex media. In particular, the boundary of interest here is initially assumed to be spherical so that

$$x^2 + y^2 + (z' + R)^2 = R^2. \quad (5.108)$$

This may be rewritten as

$$z' = -R \left[1 - \left(1 - \frac{x^2 + y^2}{R^2} \right)^{1/2} \right]. \quad (5.109)$$

Typically the extent of the beam is much smaller than the spherical radius of the boundary, and Eq. (5.109) can approximately be written

$$z' \approx -\frac{x^2}{2R_x} - \frac{y^2}{2R_y}, \quad (5.110)$$

where the boundary is also allowed to be approximately paraboloidal. In writing Eq. (5.108) it has been assumed that the boundary was aligned perpendicular to the axis. More generally, it may be assumed that the boundary is rotated in the xz -plane and translated with respect to x . In this case, an alternate coordinate system is of interest:

$$\begin{bmatrix} z' \\ x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x_o \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta_x & \sin \theta_x & 0 & 0 \\ -\sin \theta_x & \cos \theta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z'' \\ x'' \\ y'' \\ 1 \end{bmatrix}, \quad (5.111)$$

$$\begin{bmatrix} z'' \\ x'' \\ y'' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_x & -\sin\theta_x & 0 & x_o \sin\theta_x \\ \sin\theta_x & \cos\theta_x & 0 & -x_o \cos\theta_x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z' \\ x' \\ y' \\ 1 \end{bmatrix}. \quad (5.112)$$

In this coordinate system the boundary is aligned, and thus

$$z'' = -\frac{x''^2}{2R_x} - \frac{y''^2}{2R_y}. \quad (5.113)$$

Combining Eqs. (5.110), (5.112) and (5.113) results in

$$z' = -\frac{x^2}{2R_x} \cos\theta_x + \left[\tan\theta_x + \frac{x_o \cos\theta_x}{R_x} \right] x - \frac{y^2}{2R_y \cos\theta_x} - \left[\frac{x_o^2 \cos\theta_x}{2R_x} + x_o \tan\theta_x \right] \quad (5.114)$$

where the approximation

$$\left| \frac{(x - x_o) \sin\theta_x}{R_x} \right| \ll 1 \quad (5.115)$$

has been used to avoid twisting of the input beam and higher-order aberrations. Now Eq.

(5.114) may be combined with Eq. (5.97), and after some algebra it follows that

$$T_{spherical \ boundary, x} = \begin{bmatrix} 1 & 0 & 0 \\ (1 - k_{o1}/k_{o2}) R_x^{-1} \cos\theta_x & k_{o1}/k_{o2} & 0 \\ (k_{o1} - k_{o2})(\tan\theta_x + x_o R_x^{-1} \cos\theta_x) & 0 & 1 \end{bmatrix}, \quad (5.116)$$

$$T_{spherical \ boundary, y} = \begin{bmatrix} 1 & 0 & 0 \\ (1 - k_{o1}/k_{o2})(R_y \cos\theta_x)^{-1} & k_{o1}/k_{o2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (5.117)$$

$$P_2 = P_1 - x_o^2 \frac{\cos\theta_x}{2R_x} - x_o \tan\theta_x. \quad (5.118)$$

The beam matrices for a thin lens may be derived by combining two spherical boundary matrices in the usual way, and it follows that

$$T_{thin\ lens,x} = \begin{bmatrix} 1 & 0 & 0 \\ -f_{active,x}^{-1} \cos \theta_x & 1 & 0 \\ -\beta_o x_o f_{active,x}^{-1} \cos \theta_x & 0 & 1 \end{bmatrix}, \quad (5.119)$$

$$T_{thin\ lens,y} = \begin{bmatrix} 1 & 0 & 0 \\ -(f_{active,x} \cos \theta_x)^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (5.120)$$

where

$$f_{active,x} \equiv (1 - \beta_{air}^{-1} k_{o,lens})(R_x^{-1} - R_x^{-2}) . \quad (5.121)$$

The quantity $k_{o,lens} = 2\pi n_{o,lens} \lambda^{-1} + i \alpha_{o,lens}$ is the complex propagation constant of the lens material, and it may be noted that $f_{active,x}$ is therefore complex. When the lens is made of a lossless dielectric, or when the low gain (or loss) per wavelength approximation is made, $f_{active,x}$ becomes the usual focal length f_x . The results of this subsection are summarized in the second half of Table VII.

Discussion

With the results from the previous section, one may model the propagation of a Gaussian beam's spotsize, phase front curvature, position, and slope through an optical system in which the elements may be misaligned and nonlossless. The general procedure is begun by finding the input beam parameters Q_x and Q_y from Eq. (5.49) and the input displacement parameters S_x and S_y from Eq. (5.53). However, in the usual case, the beam is input in a homogeneous medium such as freespace, and the much simpler Eq. (5.54) may be used instead of Eq. (5.53). The system matrix may be obtained by multiplying the matrix representation for each of the optical elements in the reverse of the order they are encountered by the beam. These matrix representations are given in Tables VI and VII. Once the system matrix and the input beam parameters have been

obtained, the output beam parameters may be found from the Kogelnik Transformation Eq. (5.72). Similarly, the output displacement parameters may be obtained from Eq. (5.73). From the output beam parameters, the output spotsize and phase curvature may be found from Eq. (5.49). Finally, the beam's output position and slope may be obtained from Eq. (5.54).

The above procedure may be made more transparent for those familiar with Gaussian beam matrix theory if the following analogies are noted:

Q_x	\longleftrightarrow	S_x
$Q_x = \frac{\beta_o}{R_x} - i \frac{2}{w_x^2}$	\longleftrightarrow	$S_x = -Q_x d_{xa} + \beta_o d'_{xa}$
$\frac{Q_{x2}}{k_{o2}} = \frac{C_x + D_x Q_{x1}/k_{o1}}{A_x + B_x Q_{x1}/k_{o1}}$	\longleftrightarrow	$S_{x2} = \frac{S_{x1}}{A_x + B_x Q_{x1}/k_{o1}} + \frac{G_x + H_x Q_{x1}/k_{o1}}{A_x + B_x Q_{x1}/k_{o1}}$
Thin Lens	\longleftrightarrow	Thin Prism
Gaussian Aperture	\longleftrightarrow	Exponential Aperture
Spherical Mirror	\longleftrightarrow	Tilted Flat Mirror
Complex Lenslike Medium	\longleftrightarrow	Complex Prismlike Medium .

There is no analytical matrix representation for a complex lenslike medium which is arbitrary tapered and curved. However, it is possible to obtain the matrix representation for complex lenslike media with several specific tapering and curvature functions. To obtain the matrix representation for such an optical element, the complete complex propagation constant must first be found. In general, the curvature of the gain axes [Eqs. (5.43) and (5.44)] may be different from the curvature of the index of refraction axes [Eqs. (5.41) and (5.42)]. In any case, Eq. (5.40) is useful to obtain the propagation constant. Once the propagation constant is known, differential Eqs. (5.56) and (5.57) must be solved. If exact analytic solutions [79] [132] are not readily obtained, WKB approxi-

mation techniques [79], solution-generating methods [132], series expansions, piecewise approximation techniques [81], propagation constant approximations [81], or numerical methods [78] may be used. The solutions to Eq. (5.56) must be put into the form of Eq. (5.58) to obtain A_x , A_y , B_x , and B_y . The remaining elements G_x , G_y , H_x , and H_y are obtained from the integrals in Eqs. (5.64) and (5.65).

The matrices in Tables VI and VII are also generalized from previous similar tables of beam matrices in that the possibility of media with high nonsaturating gain per wavelength are included. Thus the complex propagation constant k_o ($k_o = \beta_o + i\alpha_o$) is used instead of the symbol n for index of refraction, as is common.

As an example of the formalism, the transmission characteristics of an exponential aperture having the matrix shown in Table VII are examined. Because of the simplicity of the matrix, Eqs. (5.49) and (5.54) may be combined into Eqs. (5.83) and (5.84), which immediately reduce to

$$d_{xa2} = d_{xa1} + \frac{w_x^2}{2w_{ea,x}} \quad (5.122)$$

$$d'_{xa2} = d'_{xa1} + \frac{w_x^2}{2R_x w_{ea,x}} \quad (5.123)$$

where $w_{ea,x}$ is the damping width of the exponential aperture. A beam's spotsize and phase curvature are unaffected by the aperture. Similar to the Gaussian aperture, the exponential aperture may be used for beam diagnostics by only measuring beam positions and slopes. Of particular interest is that for an exponential aperture

$$\frac{d_{xa2} - d_{xa1}}{d'_{xa2} - d'_{xa1}} = R_x \quad (5.124)$$

Though this relation is valid for an exponential aperture, it follows from Eqs. (5.81) and (5.82) that this same relation applies to Gaussian apertures as well.

SUMMARY

All Gaussian beam optical systems are, to some extent, nonlossless. Similarly, tilts and misalignments within the system are inevitable. Furthermore, in modern optical systems design, these effects are often not accidental, but are introduced for system enhancement. For example, high diffraction loss ("unstable") resonators use transverse losses to achieve high power in a single highly stable mode, and prism pairs are used for chirp compensation. It is therefore useful to obtain a systematic methodology to analyze these systems.

A novel transfer matrix formalism has been developed from first principles which models the propagation of Gaussian beams of light in misaligned, nonlossless optical systems. The optical systems are represented by complex 3×3 generalized Gaussian beam matrices, and matrices have been developed for several optical elements. Analogies have been used for nomenclature, and have suggested new optical elements such as the exponential variable reflectivity mirror and the complex prismlike medium.

CHAPTER VI

LASER RESONATOR MODES

INTRODUCTION

One of the most important applications of the Gaussian beam theory in the previous chapter involves the prediction of the output beam spotsize and radius of phase front curvature of a laser oscillator. However, in the last chapter the output beam profile was assumed to be Gaussian only by postulate. In this chapter, it is shown that there are other laser oscillator eigenmodes, and stability criteria are used to determine which eigenmode a given laser operates in.

FUNDAMENTAL GAUSSIAN MODE

The Gaussian mode of a resonator that contains only complex optical elements that have generalized beam matrix representations may be found from the self-consistency requirement that q_x repeat after a round trip through the resonator. If the round-trip matrix elements are A_x , B_x , C_x , D_x , G_x , and H_x , then the steady-state beam parameter $q_{x\infty}$ at the chosen reference plane in the chosen direction must be a solution of

$$1/q_{x\infty} = \frac{C_x + D_x/q_{x\infty}}{A_x + B_x/q_{x\infty}} . \quad (6.1)$$

If $B_x \neq 0$, then this is a quadratic equation having the two roots

$$1/q_{x\infty} = \frac{D_x - A_x}{2B_x} \pm \frac{i}{B_x} \left[1 - \left(\frac{A_x + D_x}{2} \right)^2 \right]^{1/2} , \quad (6.2)$$

where the unimodularity condition $A_x D_x - B_x C_x = 1$, which applies to all optical resonators and periodic optical systems, has been used.

Equation (6.2) is sometimes rearranged as

$$A_x + B_x/q_{x\infty} = \exp[\pm i\theta_x] , \quad (6.3)$$

where

$$\cos\theta_x \equiv \frac{A_x + D_x}{2} . \quad (6.4)$$

The two solutions given in Eq. (6.2) are not necessarily acceptable in a practical situation, and sometimes neither solution is desirable. For example, if all of the matrix elements are real and the condition $-1 \leq (A_x + D_x)/2 \leq 1$ is not satisfied, then from Eq. (6.2) the beam parameter is also real. But from Eq. (5.49) the resulting spotsize is infinite. Therefore, one may impose on Eq. (6.2) a confinement condition which requires that the square of the spotsize must be positive and finite. However, confinement in one plane of an optical resonator does not guarantee confinement everywhere within the resonator. Similarly, confinement for a given direction of propagation at a given plane does not necessarily imply confinement for the opposite direction of propagation at the same plane. Confinement at all planes within the resonator for light beams propagating in both directions is required for a "confined mode." However, there are lasers which exhibit unconfined ring modes [16].

A similar self-consistency requirement exists for the position and slope of the beam, and applying the condition $S_{x1} = S_{x2} = S_{x\infty}$ to Eq. (5.73) results in

$$S_{x\infty} = \frac{G_x + H_x/q_{x\infty}}{A_x + B_x/q_{x\infty} - 1} . \quad (6.5)$$

The resulting position and slope of the mode may be garnered from Eq. (5.53) or (5.54).

Perturbation Stability

Though physically realizable modes need not be confined, they must be perturbation stable, as perturbations due to, for example, imperfections in instrumentation and align-

ment always exist [16]. If the beam parameter is perturbed by δ'_q , then after a round trip, the perturbation becomes δ''_q . These perturbations may be related using Eq. (5.72):

$$\frac{1}{q_{x\infty}} + \delta''_q = \frac{C_x + D_x(1/q_{x\infty} + \delta'_q)}{A_x + B_x(1/q_{x\infty} + \delta'_q)} \quad (6.6)$$

$$= \left[\frac{C_x + D_x/q_{x\infty}}{A_x + B_x/q_{x\infty}} \right] \left[\frac{1 + \delta'_q D_x / (C_x + D_x/q_{x\infty})}{1 + \delta'_q B_x / (A_x + B_x/q_{x\infty})} \right] \quad (6.7)$$

$$\approx \left[\frac{C_x + D_x/q_{x\infty}}{A_x + B_x/q_{x\infty}} \right] \left[1 + \frac{\delta'_q D_x}{(C_x + D_x/q_{x\infty})} - \frac{\delta'_q B_x}{(A_x + B_x/q_{x\infty})} \right] . \quad (6.8)$$

The unimodularity condition ($A_x D_x - B_x C_x = 1$) may be used, and it follows that for a small initial perturbation δ'_q ,

$$\left| \frac{\delta''_q}{\delta'_q} \right| \approx \frac{1}{|A_x + B_x/q_{x\infty}|^2} . \quad (6.9)$$

Though it is assumed in Eq. (6.2) that $B_x \neq 0$, this result is valid even when $B_x = 0$. Mode stability is assured if the perturbation damps after each round trip. As was first shown in Ref 16, this is the case if

$$F_{xs} \equiv |A_x + B_x/q_{x\infty}| > 1 . \quad (6.10)$$

Applying this stability criterion to Eq. (6.3), it follows that the two solutions given in Eq. (6.2) have stability factors that are reciprocals of each other. Thus, when one solution is stable, the other is unstable. There is always one and only one stable solution except when $A_x + D_x$ is purely real and less than two in absolute value, in which case both solutions are metastable.

A similar stability analysis may be performed for the displacement parameter:

$$S_{x\infty} + \delta''_S = \frac{S_{x\infty} + \delta'_S}{A_x + B_x/q_{x\infty}} + \frac{G_x + H_x/q_{x\infty}}{A_x + B_x/q_{x\infty}} . \quad (6.11)$$

For a perturbation of any size,

$$\left| \frac{\delta_S''}{\delta_S'} \right| = \frac{1}{|A_x + B_x/q_{x\infty}|} = F_{xs}^{-1} . \quad (6.12)$$

Thus, the same condition for stability exists [Eq. (6.10)] for the displacement parameter S_x , as for the beam parameter, Q_x . These stability results are independent of whether the system is misaligned or curved, since the stability factor is independent of G_x and H_x .

POLYNOMIAL-GAUSSIAN MODES

With the approximations discussed in Chapter V, the eigenmodes for any medium described by Eq. (5.31) and several thin optical elements such as a lens, prism, Gaussian aperture, and exponential aperture can be written

$$\bar{E}_{mn}(x, y, z, t) = \text{Re} \left\{ E'_{mn}(x, y, z) \exp[i(\omega t - \int_0^z k_o(z') dz')] \right\} [\cos v \bar{i}_x + \sin v \bar{i}_y] , \quad (6.13)$$

where the complex electric field factor is [138]

$$E'_{mn}(x, y, z) = E'_{mn,o} \exp[-i[Q_x(z)x^2/2 + Q_y(z)y^2/2 + S_x(z)x + S_y(z)y + P(z)]] \\ \times H_m\{2^{1/2}[x - \delta_x(z)]/W_x(z)\} H_n\{2^{1/2}[y - \delta_y(z)]/W_y(z)\} , \quad (6.14)$$

and the notation $\text{Re}\{\}$ designates the real part of a complex function. Thus a given beam mode contains essentially three parts - a plane wave factor, a Gaussian factor, and a polynomial factor. It is this polynomial factor which differentiates the different beam modes.

Beam Transformations

The first transformation of the parameters in the polynomial factor of the electric field in Eq. (6.14) is [240]

$$W_{x2}^2 = W_{x1}^2 (A_x + B_x/q_{x1})^2 + 4iB_x(A_x + B_x/q_{x1})/k_{o1} . \quad (6.15)$$

In material media,

$$A_x D_x - B_x C_x = k_o(0)/k_o(z) , \quad (6.16)$$

$$C_x \equiv dA_x/dz, \quad (6.17)$$

$$D_x \equiv dB_x/dz, \quad (6.18)$$

$$G_x \equiv -\frac{1}{2} \int_0^z k_1(z') A_z(z') dz', \quad (6.19)$$

$$H_x \equiv -\frac{1}{2} \int_0^z k_1(z') B_z(z') dz', \quad (6.20)$$

and the differential equation for δ_x , the complex displacement of the polynomial term in Eq. (6.14), is [33]

$$\frac{d\delta_x}{dz} - \frac{1}{q_x(z)} \delta_x(z) = \frac{S_x(z)}{k_o(z)}. \quad (6.21)$$

It can be shown by direct substitution that the exact solution of Eq. (6.21) is

$$\delta_{x2} = \delta_{x1}(A_x + B_x/q_{x1}) + S_{x1}B_x/k_{o1} + (B_x G_x - A_x H_x)/k_{o1}. \quad (6.22)$$

This new transformation generalizes a previous result [110] to include the effects of misalignment. Though there has been an emphasis here on Hermite-Gaussian modes, the field distribution in Eq. (6.13) can be expressed in terms of Laguerre-Gaussian functions, and this new transformation [Eq. (6.22)] may also be used for off-axis Laguerre-Gaussian beam modes in misaligned complex optical systems [110].

Though our analysis has dealt with complex argument modes, there may also be interest in polynomial-Gaussian beam modes in which the arguments of the polynomials are real. As opposed to the complex-argument beam modes, these real argument modes have simple spherical phase fronts. It is shown in the appendix that the real-argument modes are a special case of the complex argument modes when the generalized beam matrix is purely real. However, real-valued beam matrices correspond to lossless optical systems (with the exception of a uniform thin lossy element such as a conventional partially reflecting uniform mirror). Thus, the effect of nonuniform loss is to distort the spherical phase front of a polynomial-Gaussian beam. Real argument modes are not

eigenmodes of complex optical systems.

The Gaussian factor in Eq. (6.14) contains a phase parameter $P(z)$. It is difficult to integrate the corresponding phase parameter equation for a medium represented by an arbitrary generalized beam matrix. However, in terms of an integral, the result is

$$\begin{aligned}
 P_2 - P_1 = & \frac{i}{2} \text{Ln}(A_x D_x - B_x C_x) - \frac{i}{2} \text{Ln}(A_x + B_x/q_{x1}) - \frac{i}{2} \text{Ln}(A_y + B_y/q_{y1}) \\
 & + \frac{i}{2} \left[m \text{Ln} \left[1 + \frac{4i}{k_{o1} W_{x1}^2} \frac{B_x}{A_x + B_x/q_{x1}} \right] + n \text{Ln} \left[1 + \frac{4i}{k_{o1} W_{y1}^2} \frac{B_y}{A_y + B_y/q_{y1}} \right] \right] \\
 & - \frac{B_x}{2k_{o1}} \frac{(S_{x1} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} - \frac{B_y}{2k_{o1}} \frac{(S_{y1} + G_y + H_y/q_{y1})^2}{A_y + B_y/q_{y1}} \\
 & + \frac{H_x}{2k_{o1}} (2S_{x1} + G_x + H_x/q_{x1}) + \frac{H_y}{2k_{o1}} (2S_{y1} + G_y + H_y/q_{y1}) \\
 & + \frac{1}{2k_{o1}} \int_0^z \left[G_x \frac{dH_x}{dz'} - H_x \frac{dG_x}{dz'} \right] dz' + \frac{1}{2k_{o1}} \int_0^z \left[G_y \frac{dH_y}{dz'} - H_y \frac{dG_y}{dz'} \right] dz' . \quad (6.23)
 \end{aligned}$$

The real part of the right-hand side of this equation is the axial phase shift experienced by an input beam after propagating through an optical system. If one is interested in the gain in the axial field magnitude, it may be more convenient to rewrite Eq. (6.23) as

$$\begin{aligned}
 \frac{\exp[-iP_2]}{\exp[-iP_1]} = & \frac{\left[1 + \frac{4i}{k_{o1} W_{x1}^2} \frac{B_x}{A_x + B_x/q_{x1}} \right]^{m/2} \left[1 + \frac{4i}{k_{o1} W_{y1}^2} \frac{B_y}{A_y + B_y/q_{y1}} \right]^{n/2}}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}} \\
 & \times \exp \left[\frac{i}{2k_{o1}} \left[\frac{S_{x1}^2 B_x}{A_x + B_x/q_{x1}} + \frac{S_{y1}^2 B_y}{A_y + B_y/q_{y1}} \right] \right] , \quad (6.24)
 \end{aligned}$$

where it is assumed for simplicity only that the optical system is aligned, and the input medium and output medium are identical.

Oscillation Conditions

In the previous section, we considered the stability of a fundamental-mode Gaussian beam. For polynomial-Gaussian beams, oscillation conditions must be applied to the

complex spotsize W_x and the complex displacement parameter δ_x . If the condition $W_{x2} = W_{x1} \equiv W_{x\infty}$ is applied to Eq. (6.15), then

$$W_{x\infty}^2 = \frac{4iB_x(A_x + B_x/q_{x\infty})/k_o}{1 - (A_x + B_x/q_{x\infty})^2} \quad (6.25)$$

$$= \mp \frac{2B/k_o}{[1 - [(A_x + D_x)/2]^2]^{1/2}}, \quad (6.26)$$

where Eqs. (6.3) and (6.4) have been used. Though the complex spotsize has some features which are similar to the beam parameter, there is no corresponding confinement condition.

A similar analysis is used for the displacement parameter governed by Eq. (6.22) and if $\delta_{x2} = \delta_{x1} = \delta_{x\infty}$ then the steady state complex displacement parameter is

$$\delta_{x\infty} = \frac{(S_{x\infty}B_x + B_xG_x - A_xH_x)/k_o}{1 - (A_x + B_x/q_{x\infty})} \quad (6.27)$$

$$= \frac{[(A_x - 1)H_x - B_xG_x](A_x + B_x/q_{x\infty})/k_o}{(A_x + B_x/q_{x\infty} - 1)^2} \quad (6.28)$$

$$= \frac{[B_xG_x - (A_x - 1)H_x]/k_o}{2[1 - (A_x + D_x)/2]}, \quad (6.29)$$

where Eqs. (6.3) and (6.4) have been used.

Perturbation Stability

To determine the conditions for stability of the complex spotsize, the steady-state value is perturbed as before,

$$W_{x\infty}^2 + \delta_W'' = (W_{x\infty}^2 + \delta_W')(A_x + B_x/q_{x\infty})^2 + 4iB_x(A_x + B_x/q_{x\infty})/k_o, \quad (6.30)$$

and the magnitude of the perturbation becomes

$$\left| \frac{\delta_W''}{\delta_W'} \right| = |A_x + B_x/q_{x\infty}|^2 = F_{xs}^2, \quad (6.31)$$

even for large perturbations. Thus, an input beam's complex spotsize approaches the

value given in Eq. (6.26) if $F_{xs} < 1$.

A perturbation of the complex displacement parameter

$$\delta_{x\infty} + \delta_{\delta}'' = (\delta_{x\infty} + \delta_{\delta}') (A_x + B_x/q_{x\infty}) + S_{x\infty} B_x/k_o + (B_x G_x - A_x H_x)/k_o, \quad (6.32)$$

is used to find the magnitude of the perturbation:

$$\left| \frac{\delta_{\delta}''}{\delta_{\delta}'} \right| = |A_x + B_x/q_{x\infty}| = F_{xs}. \quad (6.33)$$

Similar to the complex spotsize, an input beam's complex displacement parameter approaches the value given in Eq. (6.29) if $F_{xs} < 1$. As with the complex spotsize, if $F_{xs} < 1$, then the complex displacement parameter is stable even for large perturbations. However, both of these criteria are different from the stability criteria for the beam parameter and displacement parameter of the Gaussian portion of the field, which were stable when $F_{xs} > 1$. Thus, there are no conditions for which the Gaussian factor and the Hermite Factor are simultaneously stable for an arbitrary mode index.

Discussion

In one approach to beam mode analysis, an initial field configuration is assumed. This initial field is allowed to propagate through the resonator of interest many times and the limiting output field profile becomes a mode of the resonator. Alternatively, oscillation conditions may be used to determine the modes of a resonator. In this latter case, perturbation stability criteria must be used to determine if the mode is experimentally realizable. This latter approach has been developed here.

Optical systems of large aperture which may contain any combination of the following elements may be represented by a generalized beam matrix: Gaussian transmission filters (and Gaussian variable reflectivity mirrors), and exponential transmission filters (and exponential variable reflectivity mirrors), thin lenses, thin prisms, spherical mirrors, and any medium representable by Eq. (5.31) with input and output planes which may be

flat or quadratically curved, tilted, or displaced. When the effect of saturation on the beam mode is ignored, a laser composed of these optical elements may be represented by an infinite periodic "lens-waveguide." The generalized beam matrix for a unit cell of the lens-waveguide may be obtained by multiplying in reverse order the matrix representations of the optical elements which make up the unit cell. The laser that corresponds to the lens-waveguide will operate in the fundamental mode when the stability condition

$$F_{xs} = |A_x + B_x/q_{x\infty}| = \left| \frac{A_x + D_x}{2} \pm i \left[1 - \left(\frac{A_x + D_x}{2} \right)^2 \right]^{1/2} \right| > 1, \quad (6.34)$$

is satisfied. If the confinement condition, $w_x^{-2} > 0$, is also satisfied, then the fundamental mode is Gaussian. When the stability condition is not satisfied, there is no mode which is stable.

It has been noticed that TEM_{00} and the TEM_{10} modes have the same stability factor for aligned optical systems [241]. This can be seen by examination of the field distributions. The electric field distributions for these modes are

$$E'_{00}(x, y, z) = E'_{00,o} \exp[-i[Q_x(z)x^2/2 + Q_y(z)y^2/2 + S_x(z)x + S_y(z)y + P(z)]] \quad (6.35)$$

$$E'_{10}(x, y, z) = E'_{00} \frac{E'_{10,o}}{E'_{00,o}} \frac{2(x - \delta_x)}{W_{x1}(A_x + B_x/q_{xo})} \quad (6.36)$$

In particular, $E'_{10}(x, y, z)$ is independent of $W_{x2}(z)$ and thus the beam does not suffer from the instability of the transformation of the complex spotsize. If the system and input beam are aligned, then $\delta_{x2}(z) \equiv 0$ and it follows that the TEM_{00} and the TEM_{10} modes have the same stability.

To investigate which of the two modes the laser operates in, we compare the power gain per pass of each mode:

$$G_{p,mn} \equiv \frac{Power_{mn}(z)}{Power_{mn}(0)} = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E'^*_{mn}(x, y, z) E'_{mn}(x, y, z) dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E'^*_{mn}(x, y, 0) E'_{mn}(x, y, 0) dx dy} \quad (6.37)$$

From Eqs. (6.35) - (6.37), it follows that the ratio of the power gains of the two modes is

$$\frac{G_{p,00}}{G_{p,10}} = \frac{|E'_{00,o}|^2}{|E'_{10,o}|^2} \frac{|W_{x1}|^2}{w_{x1}^2} |A_x + B_x/q_{xo}|^2 . \quad (6.38)$$

If both modes have the same initial power, then

$$\frac{G_{p,00}}{G_{p,10}} = |A_x + B_x/q_{x\infty}|^2 = F_{xs}^2 , \quad (6.39)$$

and the laser will operate in the fundamental mode since it has a larger gain per pass than the TEM_{10} .

As a final note, the stability factor F_{xs} is independent of G_x and H_x . Thus, transverse mode selection within a laser is independent of misalignments or curvatures of the optical elements within the resonator. Furthermore, it is also independent of the presence of prisms or exponential apertures.

MULTIPASS MODES

In previous sections it was assumed that laser oscillators operate in single-pass modes. In this section, generalized matrix theorems are developed and used to determine whether multipass modes exist.

Sylvester's Theorem

The purpose of this subsection is to derive Sylvester's theorem for 3x3 matrices which have the same form as generalized beam matrices. It may be noted that these matrices [Eq.(2.81)] are qualitatively similar to conventional augmented linear matrices [Eq. (5.71)]. The simplest method used in Chapter III to derive Sylvester's theorem for the augmented linear matrices was to use the commutativity requirement that $T^s T = T T^s$ to obtain two equations with two unknowns. If this is done with the generalized beam matrix, then the two equations are

$$A_s G + C_s H + G_s = A G_s + D H_s + G , \quad (6.40)$$

$$B_s G + D_s H + H_s = B G_s + D H_s + H_s . \quad (6.41)$$

These equations may be solved for G_s and H_s to obtain Sylvester's theorem since A_s , B_s , C_s , and D_s are the same as those equivalent elements for a 2×2 matrix. However, it may be noticed that Eqs. (6.40) and (6.41) are the same as Eqs. (3.75) and (3.76) if

$$B \longleftrightarrow C \quad (6.42)$$

$$E \rightarrow G \quad (6.43)$$

$$F \rightarrow H \quad (6.44)$$

$$B_s \longleftrightarrow C_s \quad (6.45)$$

$$E_s \rightarrow G_s \quad (6.46)$$

$$F_s \rightarrow H_s . \quad (6.47)$$

Since E_s and F_s have been calculated previously, G_s and H_s are obtained, and the results are written in Table VIII.

TABLE VIII
GENERALIZED BEAM MATRIX THEOREMS

Description	Operation	Matrix $\cos\theta \equiv \frac{1}{2}(A + D)$ $\gamma \equiv A + D - 2 = 2(\cos\theta - 1)$
Sylvester's Theorem	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^s$	$\frac{1}{\sin\theta} \begin{bmatrix} A \sin(s\theta) - \sin[(s-1)\theta] & B \sin(s\theta) & 0 \\ C \sin(s\theta) & D \sin(s\theta) - \sin[(s-1)\theta] & 0 \\ G_s' & H_s' & \sin\theta \end{bmatrix}$
Reverse Theorem	$\leftarrow \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} \leftarrow$	$\begin{bmatrix} D & B & 0 \\ CH - DG & -(BG - AH) & 1 \end{bmatrix}$
Squared Matrix	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^2$	$\begin{bmatrix} A(A+D)-1 & B(A+D) & 0 \\ C(A+D) & D(A+D)-1 & 0 \\ (A+1)G+CH & BG+(D+1)H & 1 \end{bmatrix}$
Unit Matrix	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^1$	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}$
Square Root Matrix	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^{1/2}$	$\begin{bmatrix} (A+1)\gamma^{1/2} & B\gamma^{1/2} & 0 \\ C\gamma^{1/2} & (D+1)\gamma^{1/2} & 0 \\ \gamma^{1/2}(2\gamma+1)^{-1}[G(D+2)-CH] & \gamma^{1/2}(2\gamma+1)^{-1}[(A+2)H-BG] & 1 \end{bmatrix}$
Identity Matrix	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Inverse Square Root Matrix	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^{-1/2}$	$\begin{bmatrix} (D+1)\gamma^{1/2} & -B\gamma^{1/2} & 0 \\ -C\gamma^{1/2} & (A+1)\gamma^{1/2} & 0 \\ (2\gamma+1)^{-1}/(\gamma+1)CH - [(\gamma+1)D+1]G & (2\gamma+1)^{-1}/(\gamma+1)BG - [(\gamma+1)A+1]H & 1 \end{bmatrix}$
Inverse Matrix	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^{-1}$	$\begin{bmatrix} D & -B & 0 \\ -C & A & 0 \\ CH - DG & BG - AH & 1 \end{bmatrix}$
Inverse Squared Matrix	$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^{-2}$	$\begin{bmatrix} D(A+D)-1 & -B(A+D) & 0 \\ -C(A+D) & A(A+D)-1 & 0 \\ (A+1)G+CH & BG+(D+1)H & 1 \end{bmatrix}$
$G_s' \equiv [(A-1)\sin(s\theta) + (D-1)/\sin[(s-1)\theta] + \sin\theta]\gamma^{-1}G + [\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta]\gamma^{-1}CH$ $H_s' \equiv [\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta]\gamma^{-1}BG + [(D-1)\sin(s\theta) + (A-1)/\sin[(s-1)\theta] + \sin\theta]\gamma^{-1}H$		

Reverse Theorem

The reverse theorem is another generalized transfer matrix theorem discussed in

Chapter III. In this section the particular reverse theorem that applies to generalized beam matrices is derived.

The derivation for the reverse matrix follows similar derivations in Chapter III. The output beam is related to the input beam through the matrix equation

$$\begin{bmatrix} u_x \\ [Q_x/k_o]u_x \\ S_x u_x \end{bmatrix}_2 = T \begin{bmatrix} u_x \\ [Q_x/k_o]u_x \\ S_x u_x \end{bmatrix}_1. \quad (6.48)$$

Proceeding as in Chapter III, both sides are multiplied by T^{-1} :

$$\begin{bmatrix} u_x \\ [Q_x/k_o]u_x \\ S_x u_x \end{bmatrix}_1 = T^{-1} \begin{bmatrix} u_x \\ [Q_x/k_o]u_x \\ S_x u_x \end{bmatrix}_2. \quad (6.49)$$

Now the vectors must be situated to reflect the reverse operation. The quantities in the vectors propagate through differential Eqs. (5.33) and (5.35). For reverse propagation $z \rightarrow -z$, and these differential equations may be rewritten

$$[-Q_x(z)]^2 + k_o(z) \frac{\partial[-Q_x(z)]}{\partial(-z)} + k_o(z)k_{2x}(z) = 0, \quad (6.50)$$

$$[-Q_x(z)][-S_x(z)] + k_o(z) \frac{\partial[-S_x(z)]}{\partial(-z)} + \frac{k_o(z)k_{1x}(z)}{2} = 0. \quad (6.51)$$

Thus $Q_x \rightarrow -Q_x$ and $S_x \rightarrow -S_x$ for reverse propagation. The signal vector can now be written as a matrix multiplied by the corresponding signal vector traveling in the opposite direction so that Eq. (6.51) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_x \\ [-Q_x/k_o]u_x \\ -S_x u_x \end{bmatrix}_1 = T^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_x \\ [-Q_x/k_o]u_x \\ -S_x u_x \end{bmatrix}_2. \quad (6.52)$$

Multiplying both sides of Eq. (6.52) on the left by the appropriate matrix yields

$$\begin{bmatrix} u_x \\ [-Q_x/k_o]u_x \\ -S_x u_x \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_x \\ [-Q_x/k_o]u_x \\ -S_x u_x \end{bmatrix}_2. \quad (6.53)$$

Thus it follows from the definition of a reverse matrix [Eq. (3.4)] that

$$T_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (6.54)$$

The result of this calculation is included in Table VIII. If $AD-BC=1$, then the system represented by the matrix T is symmetric if $A = D$, $G = 0$, and $H = 0$.

Multipass Resonator Modes

A self-consistency requirement has been used to obtain the steady-state beam parameter [Eq. (6.1)] and displacement parameter [Eq. (6.5)]. However, it was previously assumed that these parameters repeat after a single round trip. In this subsection, we investigate whether there exist additional modes which repeat after multiple round trips.

If the beam parameter repeats after n round trips then

$$1/q_{x\infty,n} = \frac{C_{x,n} + D_{x,n}/q_{x\infty,n}}{A_{x,n} + B_{x,n}/q_{x\infty,n}}. \quad (6.55)$$

From Sylvester's theorem in Table VIII, $A_{x,n}$, $B_{x,n}$, $C_{x,n}$, and $D_{x,n}$ may be substituted into Eq. (6.55):

$$1/q_{x\infty,n} = \frac{C_x \sin(n\theta) + [D_x \sin(n\theta) - \sin[(n-1)\theta] - \sin\theta]/q_{x\infty,n}}{[A_x \sin(n\theta) - \sin[(n-1)\theta] - \sin\theta] + B_x \sin(n\theta)/q_{x\infty,n}}. \quad (6.56)$$

By multiplying both sides of Eq. (6.55) by the denominator of the right hand side of the equation, it can be seen that all of the θ terms cancel and Eq. (6.56) may be rewritten

$$1/q_{x\infty,n} = \frac{C_x + D_x/q_{x\infty,n}}{A_x + B_x/q_{x\infty,n}}. \quad (6.57)$$

However, comparing this equation with Eq. (6.1) it follows that $1/q_{x\infty,n} = 1/q_{x\infty,1}$. Thus, there are no additional multipass values to the beam parameter.

A similar analysis may be applied to the displacement parameter. The single-pass displacement parameter is the same as Eq. (6.5):

$$S_{x\infty,1} = \frac{G_x + H_x/q_{x\infty}}{A_x + B_x/q_{x\infty} - 1}. \quad (6.58)$$

The multipass displacement parameter is

$$S_{x\infty,n} = \frac{G_{x,n} + H_{x,n}/q_{x\infty}}{A_{x,n} + B_{x,n}/q_{x\infty} - 1} . \quad (6.59)$$

Substituting $A_{x,n}$, $B_{x,n}$, $G_{x,n}$, and $H_{x,n}$ from Sylvester's theorem in Table VIII, it follows after some algebra that

$$(S_{x\infty,n} - S_{x\infty,1})[(A_x + B_x/q_{x\infty})\sin(n\theta) - (\sin[(n-1)\theta] + \sin(\theta))] = 0 . \quad (6.60)$$

Thus, either the mode is a single pass mode (i. e. $S_{x\infty,n} = S_{x\infty,1}$) or it is a multipass mode which satisfies

$$\left[A_x \sin(n\theta) - (\sin[(n-1)\theta] + \sin(\theta)) \right] + \left[B_x \sin(n\theta) \right] / q_{x\infty} = 0 . \quad (6.61)$$

This is essentially the same result as obtained in Chapter III. In particular, if the criteria given in Eqs. (3.56) and (3.57) are met, then both of the bracketed quantities in Eq. (6.60) are zero, and thus Eq. (6.61) is satisfied. Alternately, $A_x + B_x/q_{x\infty} \equiv \exp(i\theta)$ may be used, and the bracketed quantity in Eq. (6.60) becomes

$$n\theta = 2k\pi \quad (6.62)$$

for any integer k . This is the same condition derived previously for aligned systems. Thus, θ must be real, and it follows that multipass modes are always metastable.

SUMMARY

In Chapter V, the generalized beam matrix method was established to propagate fundamental Gaussian beams in misaligned complex optical systems. Here, new transformations were applied to polynomial-Gaussian beams in these optical systems. An important application of the theory involves the investigation of the mode selection in lasers and periodic optical systems and this problem was considered here. In particular, it was found that when the stability factor $F_{xy} > 1$, the laser operates in the fundamental mode. If the confinement condition $w_x^{-2} > 0$ is also satisfied, then this mode is Gaussian. When $F_{xy} = 1$, all of the modes are metastable and the laser cannot discriminate between

the modes. When $F_{xs} < 1$, all of the modes are unstable. In practice, they encounter a system aperture or other mode mixing element which makes the beam mode nonGaussian, but also invalidates the generalized beam matrix characterization for the system. Paraxial misalignment has no effect on single-pass mode selection.

CHAPTER VII

CONCLUSION

A large and growing portion of laser optics may be treated using transfer matrix methods. These matrix methods are conceptually and mathematically simple, and they encourage an organized systems approach. There are additional advantages to using transfer matrix methods if they are studied as a group. Once one is familiar with one transfer matrix method, it is straightforward to learn another, even if it is from a completely different branch of science. One contribution of this work has been to give a novel summary of the properties shared by all or many of the transfer matrices. When matrix methods are studied in parallel, highly sought analogies become transparent. For example, one may gain insight into the physics of particle accelerators by studying light ray propagation.

Transfer matrix methods are widely used in the study of polarization optics. For coherent laser radiation, the simple yet powerful Jones calculus matrix method is used. For optical systems illuminated by partially coherent light the Mueller transfer matrix method is common employed.

Analytical methods and state-of-the-art computer software both used transfer matrix methods for thin film design. The same techniques are used for design of Fabry-Perot and other types of interferometers.

Perhaps the oldest and most important optics problems involve tracing light rays through optical systems. Paraxial ray matrices are very simple, and yet invaluable in compound lens design. The existence of third-order aberrations provides one the classic

problems in lens design. There also exists a transfer matrix method to treat these aberrations.

The most fundamental problem in laser optics involves the propagation of Gaussian laser beams. Beam matrices are used to propagate Gaussian beams through optical systems, and for laser resonator design. The output beam of a laser may be one of several polynomial-Gaussian modes, and analysis of these modes may be performed using Gaussian beam matrices as well.

There are optical systems that partially reflect light beams in a way that depends on the light's polarization. For these optical systems, it is straightforward to combine the reflection matrices and the Jones polarization matrices into a simple 4×4 matrix form. Similarly, there are optical systems requiring a hybrid of the paraxial ray matrices and the Gaussian beam matrices. Unlike the Jones-reflection hybrid, it is not obvious how to combine these matrix methods for misaligned complex optical systems. The solution of this difficulty requires a rigorous solution of Maxwell's equations, and this has been done here. The resulting 3×3 matrix method invented by the author uses matrices that have only six independent elements. The matrix representations for a large number of optical elements have also been obtained.

Future work may be concentrated in a variety of areas. Preliminary work has been performed in this thesis to analyze Gaussian pulses in optical systems with time-dependent properties. A reasonable percentage of lasers are pulsed, and the establishment of simple matrix methods to propagate these pulses would be highly desirable.

Saturation plays an important role in high gain lasers. Because of this, our beam analysis could not be used to, for example, predict laser output power. Saturation can also affect a laser beam's size and profile. It would be desirable to create a matrix method, or at least a systematic method, to analyze some or all of these effects.

Even our generalized beam matrix method is oblivious to the difference between a spherical mirror and a parabolic one. It would be desirable to obtain a transfer matrix method to treat aberrated laser beams.

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